VIII. *On the Influence of Geological Changes on the Earth's Axis of Rotation.* By

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Received October 13,—Read November 23, 1876.

THE subject of the fixity or mobility of the earth's axis of rotation in that body, and the possibility of variations in the obliquity of the ecliptic, have from time to time attracted the notice of mathematicians and geologists. The latter look anxiously for some grand cause capable of producing such an enormous effect as the glacial period. Impressed by the magnitude of the phenomenon, several geologists have postulated a change of many degrees in the obliquity of the ecliptic and a wide variability in the position of the poles on the earth; and this, again, they have sought to refer back to the upheaval and subsidence of continents.

Mr. JOHN EVANS, F.R.S., the late President of the Geological Society, in an address delivered to that Society, has recurred to this subject at considerable length. After describing a system of geological upheaval and subsidence, evidently designed to produce a maximum effect in shifting the polar axis, he asks:—"Would not such a modification of form bring the axis of figure about 15° or 20° south of the present, and on the meridian of Greenwich—that is to say, midway between Greenland and Spitzbergen? and would not, eventually, the axis of rotation correspond in position with the axis of figure?

"If the answer to these questions is in the affirmative, then I think it must be conceded that even minor elevations within the tropics would produce effects corresponding to their magnitude, and also that it is unsafe to assume that the geographical position of the poles has been persistent throughout all geological time."*

On the few occasions on which this subject has been referred to by mathematicians, the adequacy of geological changes to produce effects of such amount has been denied. Amongst others, the Astronomer Royal and Sir WILLIAM THOMSON have written briefly on the subject†, but, as far as I know, the subject has not hitherto been treated at much length.

The following paper is an attempt to answer the questions raised by Mr. EVANS; but as I have devoted a section to the determination of the form of continent and sea which would produce a maximum effect in shifting the polar axis, I have not taken into consideration the configuration proposed by him.

The general plan of this paper is to discuss the following problems:—

* Quart. Journ. Geol. Soc. 1876, xxxii. Proc. p. 108.

† In papers referred to below.

First. The precession and nutation of a body slowly changing its shape from internal causes, with especial reference to secular alterations in the obliquity of the ecliptic.

Second. The changes in the position of the earth's axis of symmetry, caused by any deformations of small amount.

Third. The modifications introduced by various suppositions as to the nature of the internal changes accompanying the deformations.

In making numerical application of the results of the previous discussions to the case of the earth, it has of course been necessary to betake one's self to geological evidence; but the vagueness of that evidence has precluded any great precision in the results.

In conclusion I must mention that, since this paper has been in manuscript, Sir WILLIAM THOMSON, in his Address to the Mathematical Section of the British Association at Glasgow, has expressed his opinion on this same subject. He there shortly states results in the main identical with mine, but without indicating how they were arrived at.

The great interest which this subject has recently been exciting both in England and America, coupled with the fact that several of my results are not referred to by Sir WILLIAM THOMSON, induces me to persist in offering my work to the Royal Society.

I. PRECESSION OF A SPHEROID SLOWLY CHANGING ITS SHAPE.

I begin the investigation by discussing the precession and nutation of an ellipsoid of revolution slowly and uniformly changing its shape. The changes are only supposed to continue for such a time, that the total changes in the principal moments of inertia are small compared to the difference between the greatest and least moments of inertia of the ellipsoid in its initial state.

For brevity, I speak of the ellipsoid as the earth; and shall omit some parts of the investigation, which are irrelevant to the problem under discussion.

The changes are supposed to proceed from internal causes, and to be any whatever; and in the application made they will be supposed to go on with a uniform velocity.

1. *The Equations of Motion.*

M. LIOUVILLE has given the equations of motion about a point of a body which is slowly changing its shape from internal causes*; these equations, he says, are only applicable to the case of the point being fixed or moving uniformly in a straight line. They may, however, be extended to the motion of the earth about its centre of inertia, because the centrifugal force due to the orbital motion and the unequal orbital motion will not add any thing to the moments of the impressed forces. These equations are, in fact, an extension of EULER's equations for the motion of a rigid body, which are ordinarily applied to the precessional problem. To make them intelligible I reproduce the following from Mr. ROUTH's 'Rigid Dynamics'†, where the proof is given more succinctly than in the original:—

* LIOUVILLE's Journ. Math. 2[me] série, t. iii. 1858, p. 1.

† Page 150, edit. of 1860, but omitted in later editions. L, M, N are the couples of the impressed forces about the axes.

"Let x, y, z be the coordinates of any particle of mass m at the time t, referred to axes fixed in space. Then we have the equation of motion

$$\Sigma m\left(x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}\right)=\text{N} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and two similar equations.

"Let

$$h_3=\Sigma m\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

with similar expressions for h_1, h_2.

"Then the equation (1) becomes

$$\frac{dh_3}{dt}=\text{N} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

"Let the motion be referred to three rectangular axes Ox', Oy', Oz' moving in any manner about the origin O. Let α, β, γ be the angles these three axes make with the fixed axis of z. Now h_3 is the sum of the products of the mass of each particle into twice the projection on the plane of xy of the area of the surface traced out by the radius vector of that particle drawn from the origin. Let h'_1, h'_2, h'_3 be the corresponding 'areas' described on the planes $y'z'$, $z'x'$, $x'y'$ respectively. Then by a known theorem proved in Geometry of Three Dimensions, the sum of the projections of h_1', h'_2, h'_3 on xy is equal to h_3;

$$\therefore\ h_3=h'_1\cos\alpha+h'_2\cos\beta+h'_3\cos\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

"Since the fixed axes are quite arbitrary, let them be taken so that the moving axes are passing through them at the time t. Then

$$h'_1=h_1,\ \ h'_2=h_2,\ \ h'_3=h_3;$$

and by the same reasoning, as in Arts. 114 and 115, we can deduce from equation (4) that

$$\frac{dh_3}{dt}=\frac{dh'_3}{dt}-h'_1\theta_2+h'_2\theta_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where θ_1, θ_2, θ_3 are the angular velocities of the axes with reference to themselves. Hence the equations of motion of the system become

$$\left.\begin{aligned}\frac{dh_1}{dt}-h_2\theta_3+h_3\theta_2&=\text{L},\\ \frac{dh_2}{dt}-h_3\theta_1+h_1\theta_3&=\text{M},\\ \frac{dh_3}{dt}-h_1\theta_2+h_2\theta_1&=\text{N},\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

"These equations may be put under another form which is more convenient. Let x', y', z' be the coordinates of the particle m referred to the moving axes, and let

$$\text{H}_3=\Sigma m\left(x'\frac{dy'}{dt}-y'\frac{dx'}{dt}\right).$$

"Since the fixed axes coincide with these at the time t, we have $x=x'$, $y=y'$, and by Art. 114,

$$\left.\begin{aligned}\frac{dx}{dt}&=\frac{dx'}{dt}+\theta_2 z-\theta_3 y\\ \frac{dy}{dt}&=\frac{dy'}{dt}+\theta_3 x-\theta_1 z\end{aligned}\right\};$$

$$\therefore\ h'_3=\mathrm{H}_3+\mathrm{C}\theta_3-\mathrm{E}\theta_1-\mathrm{D}\theta_2\,*;$$

and by similar reasoning

$$h'_1=\mathrm{H}_1+\mathrm{A}\theta_1-\mathrm{F}\theta_2-\mathrm{E}\theta_3,$$
$$h'_2=\mathrm{H}_2+\mathrm{B}\theta_2-\mathrm{D}\theta_3-\mathrm{F}\theta_1.$$

"Hence the general equation of motion becomes

$$\frac{d}{dt}(\mathrm{C}\theta_3-\mathrm{E}\theta_1-\mathrm{D}\theta_2+\mathrm{H}_3)+\mathrm{F}(\theta_2^2-\theta_1^2)+(\mathrm{B}-\mathrm{A})\theta_1\theta_2+\mathrm{E}\theta_2\theta_3-\mathrm{D}\theta_1\theta_3+\theta_1\mathrm{H}_2-\theta_2\mathrm{H}_1=\mathrm{N}\quad . \quad (7)$$

and two similar equations.

"Let the moving axes be so chosen as to coincide with the principal axes at the time t. Then D=0, E=0, F=0, and the equations become,"

$$\frac{d}{dt}(\lambda_1\theta_1+\mathrm{H}_1)-(\lambda_2-\lambda_3)\theta_2\theta_3+\theta_2\mathrm{H}_3-\theta_3\mathrm{H}_2=\mathrm{L},$$

and two similar equations; where λ_1, λ_2, λ_3 (replacing the A, B, C of Mr. ROUTH) are the three principal moments of inertia, and are functions of the time.

In order to apply these equations to the present problem, we must consider the meaning of the quantities θ_1, θ_2, θ_3. A system of particles may be made to pass from any one configuration to any other by means of the rotation of the system as a whole about any axis through any angle, and a subsequent displacement of every particle in a straight line to its ultimate position. Of all the axes and all the angles about and through which the preliminary rotation may be made, there is one such that the sum of the squares of the subsequent paths is a minimum. By analogy with the method of least squares this rotation may be said to be that which most nearly represents the passage of the system from one configuration to the other. If the two configurations differ by little from one another, and if the best representative rotation be such that the curvilinear path of any particle is large compared to its subsequent straight path, the system may be said to be rotating as a rigid body, and at the same time slowly changing its shape. Now this is the case we have to consider in a slow distortion of the earth.

Divide the time into a number of equal small intervals τ, and in the first interval let the earth be rigid, and let each pair of its principal axes rotate about the third (with angular velocities double those with which they actually rotate). At the end of that interval suppose that each pair has rotated about the third through angles $2\omega_1\tau$, $2\omega_2\tau$,

* A, B, C, D, E, F are, as usual, the moments and products of inertia.

$2\omega_3\tau$. Then reduce the earth to rest, and during the next interval let the matter constituting the earth flow (with velocities double those with which it actually flows) so that the pairs of principal axes have, at the end of the interval, rotated with respect to the third ones through the angles $-2\alpha\tau$, $-2\beta\tau$, $-2\gamma\tau$. Lastly let $2\theta_1\tau$, $2\theta_2\tau$, $2\theta_3\tau$ be the rotations of each pair of axes about the third by which they could have been brought directly from their initial to their final positions in the time 2τ.

Therefore, by the principle of superposition of small motions,

$$\theta_1=\omega_1-\alpha,\ \theta_2=\omega_2-\beta,\ \theta_3=\omega_3-\gamma.$$

Now supposing these two processes to go on simultaneously with their actual velocities, instead of in alternate intervals of time with double velocities, it is clear that θ_1, θ_2, θ_3 are "the angular velocities of the axes with reference to themselves"; ω_1, ω_2, ω_3 are the component angular velocities of the earth considered as a rigid body; and $-\alpha$, $-\beta$, $-\gamma$ are the component angular velocities of the principal axes relatively to the earth, arising from the supposing continuous distortion of that body.

With respect to the other quantities involved in the equations of motion:—

Let C, A be the principal moments of inertia of the earth initially when t is zero; and at any time t, let

$$\lambda_1=\mathrm{A}+\mathrm{a}t,\ \lambda_2=\mathrm{A}+\mathrm{b}t,\ \lambda_3=\mathrm{C}+\mathrm{c}t.$$

We here suppose that the changes in the earth are so slow that terms depending on higher powers of t may be neglected.

Lastly the quantities H_1, H_2, H_3 are respectively twice the areas conserved on the planes of $\theta_2\theta_3$, $\theta_3\theta_1$, $\theta_1\theta_2$ by the motion of the earth relative to these axes. If the earth were rigid, they would all be zero, because there would be no motion relative to the principal axes: thus ω_1, ω_2, ω_3 do not enter into these quantities. Now the motion which *does* take place may be analyzed into two parts. Divide the time into a number of equal small elements τ, and in the first of them let the matter constituting the earth flow (with a velocity double that with which it actually flows); this motion will conserve double-areas on the planes of $\theta_2\theta_3$, $\theta_3\theta_1$, $\theta_1\theta_2$, which we may call $2\mathfrak{H}_1\tau$, $2\mathfrak{H}_2\tau$, $2\mathfrak{H}_3\tau$. In the next interval of time let each pair of axes rotate round the third (with angular velocities double those with which they actually rotate), so that at the end of the interval they have turned through the angles $-2\alpha\tau$, $-2\beta\tau$, $-2\gamma\tau$. Now since during this second interval the axes have rotated in a negative direction through the solid, therefore the solid has rotated in a positive direction with reference to the axes. Remembering then that λ_1, λ_2, λ_3 are the principal moments of inertia, the double-areas conserved on the three planes in this second interval are $2\lambda_1\alpha\tau$, $2\lambda_2\beta\tau$, $2\lambda_3\gamma\tau$. Hence if $2\mathrm{H}_1\tau$, $2\mathrm{H}_2\tau$, $2\mathrm{H}_3\tau$ be the double areas conserved in this double interval of time, we have $2\mathrm{H}_1\tau=2\mathfrak{H}_1\tau+2\lambda_1\alpha\tau$, $2\mathrm{H}_2\tau=2\mathfrak{H}_2\tau+2\lambda_2\beta\tau$, $2\mathrm{H}_3\tau=2\mathfrak{H}_3\tau+2\lambda_3\gamma\tau$.

Therefore if we now suppose the two processes to go on simultaneously with their actual velocities, instead of in alternate elements of time with double velocities, and if we substitute for λ_1, λ_2, λ_3 their values in terms of A, a, t, &c., we get

$$H_1=(A+at)\,\alpha+\mathfrak{H}_1,\quad H_2=(A+bt)\beta+\mathfrak{H}_2,\quad H_3=(C+ct)\gamma+\mathfrak{H}_3,$$

where $\mathfrak{H}_1$, $\mathfrak{H}_2$, $\mathfrak{H}_3$, denote those parts of the double areas conserved, which depend only on the internal motions accompanying the change of shape.

Then, if the changes proceed with uniform velocity, a, α, $\mathfrak{H}$, &c. are all constant.

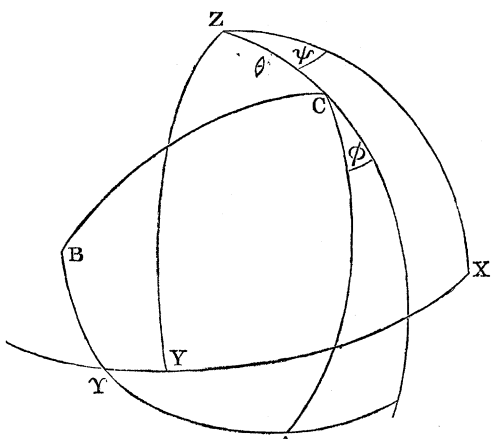
Corresponding also to the equations of motion are the geometrical equations

$$\frac{d\theta}{dt}=\theta_2\cos\phi+\theta_1\sin\phi\,;$$

$$\frac{d\psi}{dt}\sin\theta=-\theta_1\cos\phi+\theta_2\sin\phi\,;$$

$$\frac{d\phi}{dt}+\frac{d\psi}{dt}\cos\theta=\theta_3.$$

Fig. 1.



In figure 1, A, B, C are the axes, about which the moments of inertia are λ_1, λ_2, λ_3; XY is the ecliptic; and the meaning of the other symbols is sufficiently indicated.

Substituting, then, for the various symbols in the original equations of motion, it will be found that

$$A\frac{d\omega_1}{dt}-(A'-C)\omega_2\omega_3=L-t\left\{a\frac{d\omega_1}{dt}-(b-c)\,\omega_2\omega_3+b\gamma\omega_2-c\beta\omega_3\right\}-a\omega_1-\{A'\gamma+\mathfrak{H}_3\}\omega_2$$
$$+\{C\beta+\mathfrak{H}_2\}\omega_3+\beta\mathfrak{H}_3-\gamma\mathfrak{H}_2,$$

and two similar equations*.

Now the terms on the right-hand side are always very small compared to $A\frac{d\omega_1}{dt}$, because the time will not run on until they have become large; hence approximate values may be substituted therein.

* The A is written A′ in two places, where it may be taken to stand for B; and then the other equations may be found by cyclic changes of letters and suffixes.

Let the angular velocity of rotation of the earth be $-n$, and let Π cosec θ be the precession of the equinoxes; then in the small terms the following substitutions may be made:—

$$\phi=-(nt+e),\ \omega_1=-\Pi\cos(nt+e),\ \omega_2=-\Pi\sin(nt+e),\ \omega_3=-n,$$

and the e may be omitted for brevity.

Further, N (depending on the attractions of the sun and moon) is very small; and a consideration of the third equation of motion shows that, when integrated, it leads to $\omega_3=-n+$ terms, which are very small during the limited period under consideration. And if these terms were substituted on the left-hand side of the two former equations, they would be still further diluted by multiplication by the small quantities $\frac{\mathrm{C}-\mathrm{A}}{\mathrm{A}}$ and Π. Hence the third equation may be neglected, and in the two former equations $-n$ may be substituted for ω_3.

Making these substitutions, then, and writing μ for $\frac{\mathrm{C}-\mathrm{A}}{\mathrm{A}}n$, the equations become

$$\frac{d\omega_1}{dt}-\mu\omega_2=\frac{\mathrm{L}}{\mathrm{A}}-\frac{\Pi n}{\mathrm{A}}\left\{\mathrm{a}-\mathrm{b}+\mathrm{c}-\frac{\mathrm{b}\gamma}{n}\right\}t\sin nt+\frac{\Pi\mathrm{a}}{\mathrm{A}}\cos nt+\Pi\left\{\gamma+\frac{\mathfrak{H}_3}{\mathrm{A}}\right\}\sin nt$$
$$-\beta\left\{\frac{(\mathrm{C}+ct)n-\mathfrak{H}_3}{\mathrm{A}}\right\}-\frac{\mathfrak{H}_2}{\mathrm{A}}(n+\gamma);$$

$$\frac{d\omega_2}{dt}+\mu\omega_1=\frac{\mathrm{M}}{\mathrm{A}}-\frac{\Pi n}{\mathrm{A}}\left\{-\mathrm{a}+\mathrm{b}+\mathrm{c}-\frac{\mathrm{a}\gamma}{n}\right\}t\cos nt+\frac{\Pi\mathrm{b}}{\mathrm{A}}\sin nt-\Pi\left(\gamma+\frac{\mathfrak{H}_3}{\mathrm{A}}\right)\cos nt$$
$$+\alpha\left\{\frac{(\mathrm{C}+ct)n-\mathfrak{H}_3}{\mathrm{A}}\right\}+\frac{\mathfrak{H}_1}{\mathrm{A}}(n+\gamma).$$

Then neglecting $\frac{\gamma}{n}$ compared to unity, putting $\mathrm{C}+ct=\mathrm{A}$ in the small terms, and only retaining the more important terms,

$$\frac{d\omega_1}{dt}-\mu\omega_2=\frac{\mathrm{L}}{\mathrm{A}}-\frac{\Pi n}{\mathrm{A}}\{\mathrm{a}-\mathrm{b}+\mathrm{c}\}t\sin nt+\frac{\Pi\mathrm{a}}{\mathrm{A}}\cos nt+\Pi\left\{\gamma+\frac{\mathfrak{H}_3}{\mathrm{A}}\right\}\sin nt-n\beta,$$

$$\frac{d\omega_2}{dt}+\mu\omega_1=\frac{\mathrm{M}}{\mathrm{A}}+\frac{\Pi n}{\mathrm{A}}\{-\mathrm{a}+\mathrm{b}+\mathrm{c}\}t\cos nt+\frac{\Pi\mathrm{b}}{\mathrm{A}}\sin nt-\Pi\left\{\gamma+\frac{\mathfrak{H}_3}{\mathrm{A}}\right\}\cos nt+n\alpha^{*}.$$

These are the required equations of motion, and in integrating them they may be treated as linear.

* If we wish to treat a, α, $\mathfrak{H}_1$, &c. as variable, we have only to add to the right-hand sides of these equations $\frac{\Pi}{\mathrm{A}}t\cos nt\frac{d\mathrm{a}}{dt}-\frac{1}{\mathrm{A}}\frac{d\mathfrak{H}_1}{dt}$, and $\frac{\Pi}{\mathrm{A}}t\sin nt\frac{d\mathrm{b}}{dt}-\frac{1}{\mathrm{A}}\frac{d\mathfrak{H}_2}{dt}$ respectively. If we put $\mathrm{L}=\mathrm{M}=0$ and neglect $\mathfrak{H}_1$, $\mathfrak{H}_2$, $\mathfrak{H}_3$, these equations will be found to be identical with the equations (2) given by Sir W. THOMSON in App. C. I had not noticed until it was pointed out by him, how nearly applicable my equations were to the case of varying velocities of distortion.—April 26, 1877.

2. *Inequalities independent of the Impressed Forces.*

First, then, suppose that $L=M=0$.

Integrate the equations, and neglect μ compared with n, and we have

$$\omega_1=\frac{\Pi}{A}(a-b+c)\,t\cos nt+\frac{\Pi}{An}(b-c)\sin nt-\frac{\Pi}{n}\left(\gamma+\frac{\mathfrak{H}_3}{A}\right)\cos nt$$
$$+\frac{n}{\mu}\left(\alpha+\frac{\mathfrak{H}_1}{A}\right)+F\cos\mu t-G\sin\mu t,$$

$$\omega_2=\frac{\Pi}{A}(-a+b+c)\,t\sin nt+\frac{\Pi}{An}(c-a)\cos nt-\frac{\Pi}{n}\left(\gamma+\frac{\mathfrak{H}_3}{A}\right)\sin nt$$
$$+\frac{n}{\mu}\left(\beta+\frac{\mathfrak{H}_2}{A}\right)-F\sin\mu t-G\cos\mu t.$$

The last two terms in ω_1 and ω_2 represent the complementary function; and the values of F and G must be determined from the initial conditions.

Now it will be shown later that α, β, γ are comparable with $\frac{a}{C-A}, \frac{b}{C-A}, \frac{c}{C-A}$; hence the terms in the second lines are much more important than those in the first. Thus in determining the values of F and G we may neglect the first lines.

Now, initially, the instantaneous axis coincides with the axis of greatest moment of inertia; so that when $t=0$, $\omega_1=\omega_2=0$, and therefore

$$F=-\frac{n}{\mu}\left(\alpha+\frac{\mathfrak{H}_1}{A}\right),\quad G=\frac{n}{\mu}\left(\beta+\frac{\mathfrak{H}_2}{A}\right).$$

The terms in F and G represent an inequality of 306 days period.

3. *The Inequality of* 306-*Days Period**.

I have worked out the value of $\mathfrak{H}_1$ and $\mathfrak{H}_2$ in two supposed cases of elevation, under certain suppositions as to the nature of the internal movements of the earth. In one of

* I have thought it necessary to discuss this inequality fully, both on account of its intrinsic interest, and because it has been referred to by the Astronomer Royal and Sir WILLIAM THOMSON.

The former says ('Athenæum,' Sept. 22, 1860):—"Now, let us suppose the earth not absolutely rigid, but that there is susceptibility to change of form, either from that degree of yielding or fracture to which most solid substances are liable, or from the hydrostatic pressure of internal fluid. This, as I conceive, puts an end to all supposition of change of axis. The first day's whirl would again make the axis of rotation to be a principal axis, and the position of the axis would then be permanent."

But Sir GEORGE AIRY is here speaking of the effect of the elevation of a mountain mass in about latitude 45°, by something like a gaseous explosion. This supposition is not at all in accordance with the belief of geologists, whereas a gradual elevation is so.

Sir W. THOMSON, on the other hand, says (Trans. Geol. Soc. Glasgow, 1874, vol. xiv. p. 312):—"In the present condition of the earth, any change in the axis of rotation could not be permanent, because the instantaneous axis would travel round the principal axis of the solid in a period of 296 days. . . . . In very early geologic ages, if we suppose the earth to have been plastic, the yielding of the surface might have made the new axis a principal axis. But certain it is that the earth at present is so rigid that no such change is possible." And he adds that practical rigidity has prevailed throughout geologic history.

them I found $\frac{\mathfrak{H}_2}{\beta A}=\frac{1}{141}$, and $\mathfrak{H}_1=0$; and in the other $\mathfrak{H}_1=\mathfrak{H}_2=0$. In order not to interrupt the thread of the argument, the calculation is given in Appendix A; it will also be more intelligible after the latter part of this paper has been read. In the general case the same kind of proportion will subsist between $\mathfrak{H}_1$ and $A\alpha$, $\mathfrak{H}_2$ and $A\beta$, and we may therefore, without serious error, neglect the former compared with the latter.

Thus, as far as concerns the present inequality,

$$\omega_1=\frac{n\alpha}{\mu}(1-\cos\mu t)-\frac{n\beta}{\mu}\sin\mu t,$$

$$\omega_2=\frac{n\alpha}{\mu}\sin\mu t+\frac{n\beta}{\mu}(1-\cos\mu t).$$

On account of this inequality the greatest angular distance (in radians) of the instantaneous axis from the pole is $\frac{2\sqrt{\alpha^2+\beta^2}}{\mu}$. It will appear from the latter part of this paper that, if the elevation of a large continent proceeds at the rate of two feet in a century, $\sqrt{\alpha^2+\beta^2}$ may be about $\frac{1}{100}''$ per annum, and μ is 360° in 306 days; whence it follows that the greatest angle made by the instantaneous axis with the axis of figure is comparable with $\frac{1}{376}''$, a quantity beyond the power of observation. On the score of these terms the instantaneous axis will therefore remain sensibly coincident with the axis of figure.

They will, moreover, produce no secular alteration on the obliquity of the ecliptic, nor in the precession, because they will appear as periodic in $\frac{d\theta}{dt}$ and $\frac{d\psi}{dt}\sin\theta$, with arguments n and $n\pm\mu$.

Now although this inequality is so small, it nevertheless is of interest.

If we map, on a tangent plane to the earth at its initial pole, the relative motion of the instantaneous axis and the pole of figure, we get, as the equation to the curve,

$$x=\frac{\alpha}{\mu}(1-\cos\mu t)-\frac{\beta}{\mu}\sin\mu t,$$

$$y=\frac{\alpha}{\mu}\sin\mu t+\frac{\beta}{\mu}(1-\cos\mu t).$$

If t be eliminated from these equations, we get

$$\left(x-\frac{\alpha}{\mu}\right)^2+\left(y+\frac{\beta}{\mu}\right)^2=\frac{\alpha^2+\beta^2}{\mu^2}.$$

Thus the relative motion is a circle, passing through the origin, and touching a line inclined to the axis of y at an angle arc $\tan\frac{\alpha}{\beta}$. Therefore the instantaneous axis describes a circle passing through the pole of figure every 306th day; and this circle touches the meridian, along which the axis of figure is travelling with uniform velocity, in consequence of the geological deformation of the earth.

The motion of the instantaneous axis in the earth is a prolate cycloid.

4. *Adjustments to a Form of Equilibrium.*

If the earth were a viscous fluid there is no doubt but that the pole of figure would tend to displace itself towards the instantaneous axis, whose mean position would be the centre of the circle above referred to.

But Sir WILLIAM THOMSON has shown* that the earth is sensibly rigid; and in any case the earth is not a viscous fluid, properly so called, although it may be slightly plastic.

M. TRESCA has shown that all solids are plastic under sufficiently great stresses, but that, until a certain magnitude of stress is reached, the solid refuses to flow†. Now in the case of a very small inequality like this, the stresses introduced by the want of coincidence of the instantaneous axis with the axis of figure are very small, even when at their maximum; and every 306th day they are zero. It seems, therefore, extremely improbable that the stresses can be great enough to bring the earth into what M. TRESCA calls the state of fluidity; and therefore it is unlikely that there can be any adaptation of the earth's form to a new form of equilibrium in consequence thereof.

In all the other inequalities introduced, whether arising from the first three terms above given in ω_1 and ω_2, or arising from the impressed forces, to be treated hereafter, the centre of the positions of the instantaneous axis is coincident with the pole of figure, and therefore there can hardly be any adaptation of figure eccentric to the axis of greatest moment to balance the stresses introduced by centrifugal force.

It would appear probable that, whilst a geological change is taking place, the earth is practically rigid for long periods. But as the earth comes to depart more and more from a form of equilibrium, the stresses due to the mutual gravitation of the parts, and to the rotation, increase gradually, until they are sufficiently great to cause the solid matter to flow. A rough kind of adjustment to a form of equilibrium would then take place. The existence of continents, however, shows that this adjustment does not take place by the subsidence of the upheaved part; and as this adaptation of form would be produced by an entirely different cause from that to which the upheaval was due, that upheaval would probably persist independently of the approximate adoption of a new form of equilibrium by the earth.

M. TRESCA's experiments on the punching of metals would lead one to believe that the change would take place somewhat suddenly, and would, in fact, be by an earthquake, or a succession of earthquakes. On each of these occasions the tendency would

* In his Address to the British Association, 1876, he states that the argument derived from precession (THOMSON and TAIT's Nat. Phil. p. 691) is fallacious; he adduces, however, a number of cogent arguments on this point.

† "Sur l'écoulement des Corps Solides," Mém. des Sav. tom. xviii.

be to adjust the form to one of equilibrium about the instantaneous axis. Now the principal axis λ_3 has (in consequence of the postulated deformation) been travelling along the meridian in longitude $\pi + \text{arc tan}\,\frac{\alpha}{\beta}$, measured from the plane containing λ_1 and λ_3.

The earthquake will take place when, to the stresses due to mutual gravitation, are superadded the maximum stresses due to centrifugal force; that is to say, when the instantaneous axis is at its greatest distance from λ_3, the axis of greatest moment of inertia. At the instant of the earthquake the principal axis will be moved towards the position of the instantaneous axis. And as the circle described by the instantaneous axis touches the meridian of displacement of the principal axis, therefore the principal axis will be carried by the adjustment towards the centre of the circle described by the instantaneous axis, and therefore perpendicular to the meridian of displacement.

Thus, if the adjusting earthquakes take place at long intervals, the motion of the principal axis will not deviate sensibly from continuing along the meridian, along which it would travel in consequence merely of the geological deformation. If, however, the adjustments are frequent, the path of λ_3 will diverge sensibly from the meridian along which it started. If the readjustments become infinitely frequent and infinitely small, there is a continuous flow of the matter of the earth, which is always seeking to bring back the earth's figure to one of equilibrium, from which figure it is also supposed to be continuously departing under the action of internal forces. In this state the earth may be considered as formed of a stiff viscous fluid.

According to these ideas, at each adjustment λ_1, λ_2, λ_3 will be suddenly reduced to nearly their primitive values, A, A, C; but α, β, γ depend on the rate of accession and diminution of matter at various parts of the earth, and remain constant. The only effect, then, is that each adjusting earthquake must be taken as a new epoch.

As far as I can see, it seems quite possible that the earth may be sensibly rigid to the tidally deforming influences of the sun and moon, and yet may bring itself back from any considerable departure from a form of equilibrium to approximately that form. It therefore seems worth while to consider the case of the adjustments being continuous, whilst the deformation is also continuous.

5. *Adjustments to the Form of Equilibrium continuous.*

I therefore propose to consider geometrically, but not dynamically, the paths of the instantaneous axis, and of the principal axis, where the earth is viscous and continuously deformed by internal forces. It is supposed that the velocities of flow of the matter of the earth are so small that inertia may be neglected, and that the displacements are so small that the principle of the superposition of small motions is applicable.

As before the paths of the instantaneous and principal axes may be mapped on a tangent plane to the spheroid, at the extremity of the primitive pole, the mean radius of the spheroid being taken as unity.

In consequence of the continuous deformation, the principal axis travels with a linear velocity (on the map) $-\sqrt{\alpha^2+\beta^2}$ along the meridian of longitude arc tan $\frac{\alpha}{\beta}$. Take this meridian as axis of x, and measure y, so that the angular velocity μ is from x towards y, and call $\sqrt{\alpha^2+\beta^2}$, u.

Then the principal axis λ_3 moves along the axis of x with a uniform linear velocity $-u$, and, from dynamical principles, the instantaneous axis I moves round the instantaneous position of λ_3 with a uniform angular velocity μ.

But because of the earth's viscosity, λ_3 always tends to approach I. The stresses introduced in the earth by the want of coincidence of λ_3 with I vary as λ_3I. Also the amount of flow of a viscous fluid, in a small interval of time, varies jointly as that interval and the stress. Hence the linear velocity (on the map), with which λ_3 approaches I, varies as λ_3I (equal to r suppose). Let this velocity be νr, where ν depends on the viscosity of the earth, diminishing as the viscosity increases.

Thus the principal axis describes a sort of curve of pursuit on the map; it is animated with a constant velocity $-u$ parallel to x, and with a velocity νr towards I, which rotates round it with a uniform angular velocity μ.

The motion of I, relative to λ_3, is that of a point moving with a constant velocity u parallel to x, rotating round a fixed point with a constant angular velocity μ, and moving towards that point with a velocity νr.

Let ξ, η be the relative coordinates of I with respect to λ_3, and x, y the coordinates of λ_3. Then the differential equations which give the above motions are:—

$$\frac{d\xi}{dt}=u-\nu\xi-\mu\eta, \quad \dots \quad (1)$$

$$\frac{d\eta}{dt}=-\nu\eta+\mu\xi, \quad \dots \quad (2)$$

$$\frac{dx}{dt}=-u+\nu\xi, \quad \dots \quad (3)$$

$$\frac{dy}{dt}=\nu\eta. \quad \dots \quad (4)$$

If (1) and (2) be integrated, and the constants determined so that, when $t=0$, $\xi=\eta=0$ (which expresses that initially λ_3 and I are coincident), it will be found that

$$\xi=\frac{u}{\nu^2+\mu^2}\left\{\nu(1-e^{-\nu t}\cos\mu t)+\mu e^{-\nu t}\sin\mu t\right\},$$

$$\eta=\frac{u}{\nu^2+\mu^2}\left\{\mu(1-e^{-\nu t}\cos\mu t)-\nu e^{-\nu t}\sin\mu t\right\}.$$

These give the path of I relative to λ_3. It may be seen to be a spiral curve diminishing with more or less rapidity, according as the earth is less or more viscous. If $\nu=0$, it becomes the circle, found above from the dynamical equations.

Substitute in (3) and (4) for ξ and η; integrate, and determine the constants, so that when $t=0$, $x=y=0$. It will then be found that

$$x=-\frac{u}{\nu^2+\mu^2}\left[\nu(\nu^2-\mu^2)+\mu^2 t+\frac{\nu}{\nu^2+\mu^2}\{-(\nu^2-\mu^2)e^{-\nu t}\cos\mu t+2\mu\nu e^{-\nu t}\sin\mu t\}\right],$$

$$y=\frac{u}{\nu^2+\mu^2}\left[-2\mu\nu^2+\mu\nu t+\frac{\nu}{\nu^2+\mu^2}\{2\mu\nu e^{-\nu t}\cos\mu t+(\nu^2-\mu^2)e^{-\nu t}\sin\mu t\}\right].$$

These give the path of λ_3 on the map. It may be seen to be a cycloidal curve, in which the radius of the rolling circle diminishes with more or less rapidity, according as the earth is less or more viscous.

After some time $e^{-\nu t}$ becomes very small, and the motion is steady; and then $\xi=\frac{u\nu}{\mu^2+\nu^2}$, $\eta=\frac{u\mu}{\nu^2+\mu^2}$, or I is fixed, relatively to λ_3, at a distance $\frac{u}{\sqrt{\nu^2+\mu^2}}$ from it, and on the meridian, measured from the axis of x, in longitude arc tan $\frac{\mu}{\nu}$. This point is the centre of the above-mentioned spiral curve.

If ν be very small (or the earth nearly rigid) this meridian differs by little from the axis of y. But it may be that ν is so small that $e^{-\nu t}$ has not time to become insensible before the geological changes cease. This case corresponds very nearly to the hypothesis, in the last section, of adjusting earthquakes.

If the earth be very mobile, or ν large, $\xi=\frac{u}{\nu}$, $\eta=0$.

Again, with respect to the path of λ_3, when the motion has become steady,

$$x=-\frac{u\nu(\nu^2-\mu^2)}{\nu^2+\mu^2}-\frac{\mu^2 u}{\nu^2+\mu^2}t,$$

$$y=-\frac{2\mu u\nu^2}{\nu^2+\mu^2}+\frac{\mu u\nu}{\nu^2+\mu^2}t,$$

and eliminating t, $\nu x+\mu y=-u\nu^2$.

That is to say, when the motion is steady, λ_3 moves parallel to meridian longitude $\pi-$arc tan $\frac{\nu}{\mu}$, and distant from it $\frac{u\nu^2}{\sqrt{\nu^2+\mu^2}}$ on the negative side. This straight line is the degraded form of the above-mentioned cycloidal curve.

If the earth is nearly rigid this path does not differ sensibly from the axis of x; if very mobile, it is nearly perpendicular to the axis of x, and a long way from the origin. In this last case the solution becomes nugatory, except as showing that the very small inequality of 306 days would be capable of disturbing and quite altering the path of the principal axis, as arising merely from geological changes on the surface of the earth.

In the case contemplated by the Astronomer Royal, where the elevation is explosive, u must be put equal to zero, and the constants of integration so determined, that when $t=0$, $\xi=\mathrm{R}$ suppose, and $\eta=x=y=0$. It will then be found that when the agitation has subsided, $x=\mathrm{R}\frac{\nu^2}{\mu^2}$, $y=\mathrm{R}\frac{\nu}{\mu}$, or the pole of figure will have taken up a position on one side of the meridian, along which it was initially propelled by the explosion.

It thus seems probable that during the consolidation of the earth there was a great instability in the position of the principal axis, and therefore also of the axis of rotation which followed it.

6. *Secular alteration in the obliquity of the Ecliptic, resulting from terms independent of the Impressed Forces.*

To return to the main line of the inquiry:—If the values of ω_1 and ω_2, found in sec. 2, be substituted in the geometrical equations for $\frac{d\theta}{dt}$ and $\frac{d\psi}{dt}\sin\theta$ (see sec. 1), a number of periodic terms will arise, and these terms have diurnal and semidiurnal periods, but their amplitudes are so small that they have no practical interest.

The only thing which concerns us is to inquire whether there can be any secular change in the obliquity of the ecliptic.

Select, then, only terms in $\sin nt$ in ω_1, and in $\cos nt$ in ω_2, and substitute in the geometrical equation $\frac{d\theta}{dt}=-\omega_1\sin nt+\omega_2\cos nt$, and reject periodic terms. It will then be found that

$$\frac{d\theta}{dt}=-\frac{\Pi}{2\mathrm{A}n}(\mathrm{a}+\mathrm{b}-2\mathrm{c}).$$

7. *Terms dependent on the Impressed Forces.*

It now remains to consider the effect of the impressed forces on the precession and obliquity of the ecliptic.

The equations of motion are reduced to

$$\frac{d\omega_1}{dt}+\frac{\mathrm{C}-\mathrm{A}}{\mathrm{A}}\omega_3\omega_2=\frac{\mathrm{L}}{\mathrm{A}},$$

$$\frac{d\omega_2}{dt}-\frac{\mathrm{C}-\mathrm{A}}{\mathrm{A}}\omega_3\omega_1=\frac{\mathrm{M}}{\mathrm{A}},$$

$$\frac{d\omega_3}{dt}=\frac{\mathrm{N}}{\mathrm{A}}.$$

If we write for L, M, N, L+δL, M+δM, δN, and indicate by L and M the couples caused by the attractions of the sun and moon on the protuberant parts of the earth before it has begun to change its shape, then L and M only cause the ordinary precession and nutations. For the present problem it is therefore only necessary to consider the effects of δL, δM, δN, which arise from the change of shape of the earth.

It follows, from the same arguments that were used in sec. 1, that the change in the earth's angular velocity of rotation due to δN will only have a very small effect on ω_1 and ω_2; so that, as far as is now important, ω_3 may be put equal to $-n$ in the first two equations, which may then be written

$$\frac{d\omega_1}{dt}-\mu\omega_2=\frac{\delta L}{A},$$

$$\frac{d\omega_2}{dt}+\mu\omega_1=\frac{\delta M}{A}.$$

Now δL and δM are the changes in L and M, when $A+at$, $A+bt$, $C+ct$ are written for A, A, and C respectively. If δL, δM be thus formed, and the equations integrated, it will be found that the principal terms, arising from the sun's attraction, are nine both in $\frac{d\theta}{dt}$ and $\frac{d\psi}{dt}\sin\theta$; the same number of terms arise in the precession and nutation with respect to the plane of the lunar orbit, and these would have to be referred to the ecliptic. Sixteen out of the eighteen terms represent, however, only very small nutations, and the only terms of any interest are those which give rise to a secular change in the obliquity of the ecliptic. These terms may be picked out without reproducing the long calculation above referred to, for they arise entirely out of the constant couple acting about the equinoctial line, which gives rise to the uniform precession.

Now this constant couple is $C\Pi n$; whence $L=C\Pi n\sin nt$, $M=-C\Pi n\cos nt$. And since Π involves $\frac{C-A}{C}$, therefore

$$\delta L=-C\Pi n\frac{b-c}{C-A}t\sin nt,\quad \delta M=-C\Pi n\frac{c-a}{C-A}t\cos nt.$$

If these be substituted in the equations of motion and the equations integrated, and only terms in $\sin nt$ in ω_1, and those in $\cos nt$ in ω_2, be retained, we get

$$\omega_1=-\frac{\Pi}{n}\frac{b-c}{C-A}\sin nt,\quad \omega_2=-\frac{\Pi}{n}\frac{c-a}{C-A}\cos nt.$$

Substituting in the geometrical equation $\frac{d\theta}{dt}=-\omega_1\sin nt+\omega_2\cos nt$, and rejecting periodic terms,

$$\frac{d\theta}{dt}=\frac{\Pi}{2n}\frac{a+b-2c}{C-A}.$$

8. *General result with respect to the Obliquity of the Ecliptic.*

It was found in sec. 6 that the secular rate of change of θ, as due to the internal changes in the earth, was $-\frac{\Pi}{2n}\cdot\frac{a+b-2c}{A}$. Since $C-A$ is small compared to A, this term is small compared with the term found at the end of sec. 7. Hence, finally, taking all the terms together, we get the approximate result,

$$\frac{d\theta}{dt}=\frac{\Pi}{2n}\cdot\frac{a+b-2c}{C-A},$$

and for small changes in the obliquity, insufficient to materially affect Π,

$$\theta=i+\frac{\Pi}{2n}\cdot\frac{a+b-2c}{C-A}t.$$

This equation has been obtained on the supposition that the change in the earth's form never becomes so great that at, bt, ct exceed small fractions of C—A; a condition which is satisfied in the case of such geological changes as those of which we have any cognizance at present.

It will appear from a comparison with results given hereafter, that $\frac{\mathrm{a}+\mathrm{b}-2\mathrm{c}}{\mathrm{C}-\mathrm{A}}t$ cannot ever exceed two or three degrees; and since $\frac{\Pi}{2n}$ is a very small fraction, it follows that *the obliquity of the ecliptic must have remained sensibly constant throughout geological history*. Also the instantaneous axis of rotation must always have remained sensibly coincident with the principal axis of figure, however the latter may have wandered in the earth's body.*

It has hitherto been assumed that the change of form and the angular velocities of the principal axes in the earth's body are uniform. But the preceding investigation shows clearly that no material change would be brought about by supposing the changes to proceed with varying velocities. This being so, dynamical considerations may be dismissed henceforth; and accordingly the next part of this paper will be devoted to the kinematical question, as to the change in position of the earth's axis of figure as due to geological changes.

The various assumptions made above will incidentally be justified in the course of the work.

For some remarks of Sir WILLIAM THOMSON on this part of the paper see Appendix C.

* During the Glacial Period there must have been heavy ice-caps on one or both poles of the earth. The above equation will give the disturbance of the obliquity of the ecliptic produced thereby.

I will take what I believe is the most extreme view held by any geologist. Mr. BELT is of opinion that an enormous ice-sheet, which was thickest in about lat. 70° N. and S., descended from both poles down to lat. 45°; the amount of ice was so great that the sea stood some 2000 feet lower than now throughout the unfrozen regions between lat. 45° N. and S.

Suppose that the whole of this equatorial region was sea, and that the water contained in 2000 feet of depth of this sea was gradually piled on the polar regions in the form of ice. Then the effect in diminishing C and increasing A cannot be so great as if the whole of this mass were subtracted actually from the equator and piled actually on the poles. This latter supposition will then give a superior limit to the amount of alteration in the obliquity of the ecliptic. I have calculated this alteration by means of the above formula, taking the numerical data used later in this paper, and taking the specific gravity of water to that of surface-rock as 4 to 11. I find, then, that the superior limit to the increase of the obliquity of the ecliptic would be ·00045″; that is to say, the position of the arctic circle cannot have been shifted so much as half an inch. And this is an accumulated effect, and the matter is distributed in the most favourable manner possible.

In this case the amount of matter displaced is enormous, and is placed in the most favourable position for affecting the obliquity; hence, *à fortiori*, geological changes in the earth cannot have sensibly affected the obliquity.

But although this equation leads to no startling results in the geological history of the earth, I hope to show in a future paper that it may have some bearing on the very remote history of the earth and of the other planets (see a paper "On a Suggested Explanation of the Obliquity of Planets to their Orbits," Phil. Mag. March 1877).—[In consequence of a mistake in the work it was erroneously stated in the abstract of this paper in the 'Proceedings' that the change in the position of the arctic circles might amount to 3 inches, instead of to half an inch.—*Added* August 18, 1877.]

II. THE PRINCIPAL AXES OF THE EARTH.

9. *Preliminary Assumptions.*

It is assumed at first that, in consequence of some internal causes, the earth is undergoing a deformation, but that there is no disturbance of the strata of equal density, and that there is no local dilatation or contraction in any part of the body. The cases at present excluded will be considered later.

The result of this assumption is, that the volume of the body remains constant, and that the parts elevated or depressed above or below the mean surface of the ellipsoid have the same density as the rest of the surface. Such changes of form must, of course, be produced by a very small flow of the solid matter of the earth. Since the whole volume remains the same, this hypothesis may be conveniently called that of incompressibility; although, if the matter of the earth flowed quite incompressibly, there would be some slight dislocation of the strata of equal density.

It is immaterial for the present purpose what may be the forces which produce, and the nature of, this internal flow; but it was assumed in the dynamical investigation that the forces were internal, and that the flow proceeded with uniform velocity.

After deformation the body may be considered as composed of the original ellipsoid, together with a superposed layer of matter, which is positive in some parts and negative in others. The condition of constancy of volume necessitates that the total mass of this layer should be zero. If we take axes with the origin at the centre of the ellipsoid and symmetrical thereto, and let $h\,\mathrm{F}(\theta, \phi)$ represent the depth of the layer at the point θ, ϕ, the condition of incompressibility is expressed by the integral of $\mathrm{F}(\theta, \phi)$ over the surface of the ellipsoid being zero. Then by varying h, elevations and depressions of various magnitudes may be represented.

10. *Moments and Products of Inertia after Deformation.*

Before the deformation:—

Let A, C be the principal moments of inertia of the earth; a, b its semiaxes; M its mass; $\mathfrak{D}$ its mean density; ϱ its surface density; and c its mean radius, so that $3c=2a+b$; and let the earth's centre of inertia be at the origin.

After the deformation:—

Let a, b, c, D, E, F be the moments and products of inertia of the above ideal shell of matter about the axes; $\bar{x}, \bar{y}, \bar{z}$ the coordinates of the earth's centre of inertia.

Then, since the ellipticity of the earth is small, the integrals may be taken over the surface of a sphere of radius c, instead of over the ellipsoid. Therefore,

$$\mathrm{a}=h\varrho c^4 \iint \mathrm{F}(\theta, \phi) \sin\theta\,(\sin^2\theta\sin^2\phi+\cos^2\theta)\,d\theta\,d\phi,$$

$$\mathrm{M}\bar{x}=h\varrho c^3 \iint \mathrm{F}(\theta, \phi)\sin^2\theta\cos\phi\,d\theta\,d\phi,$$

$$\mathrm{M}=\tfrac{4}{3}\pi\mathfrak{D}c^3,$$

and other integrals of a like nature for b, c, D, E, F, $\bar{y}$, $\bar{z}$.

Since

$$\iint F(\theta, \phi) \sin\theta \, d\theta \, d\phi = 0, \text{ therefore } a+b+c=0.$$

If A be the moment of inertia of the body, after deformation, about an axis parallel to x, through $\bar{x}, \bar{y}, \bar{z}$,

$$A = \mathrm{A} + \mathrm{a} - \mathrm{M}(\bar{y}^2 + \bar{z}^2).$$

Now a varies as $\frac{h}{c}$, whilst $\mathrm{M}(\bar{y}^2+\bar{z}^2)$ varies as $\left(\frac{h}{c}\right)^2$. But the greatest elevation or depression to be treated of is about two miles, whilst the mean radius c is about 4000 miles; hence $\frac{h}{c}$ cannot exceed about $\frac{1}{2000}$, and accordingly the term $\mathrm{M}(\bar{y}^2+\bar{z}^2)$ is negligible compared to a. Whence $A = \mathrm{A}+\mathrm{a}$.

In like manner, the terms introduced in the other moments and products of inertia by the shifting of the earth's centre of inertia are negligible compared to the direct changes. Thus it may be supposed that the centre of inertia remains fixed at the origin, and that the moments and products of inertia of the earth after deformation are A+a, A+b, C+c, D, E, F.

11. *General Theorem with respect to principal Axes.*

A general theorem will now be required to determine the position of the principal axes after the deformation.

Take as axes the principal axes of a body about which its moments of inertia are A, B, C. Let the body undergo a small deformation, which turns the principal axes through small angles α, β, γ about the axes of reference, and makes the new principal moments A′, B′, C′. And let the moments and products about the axes of reference become in consequence A+a, B+b, C+c, D, E, F. Then it is required to find α, β, γ in terms of these last quantities.

Let l, m, n be the direction cosines of any line through the origin, and let them remain unaltered by the deformation. Let I be the moment of inertia about this line after deformation. Let $l+\delta l$, $m+\delta m$, $n+\delta n$ be the direction cosines of the line with respect to the new principal axes. Then, by a well-known theorem,

$$\delta l = \gamma m - \beta n, \quad \delta m = \alpha n - \gamma l, \quad \delta n = \beta l - \alpha m.$$

Now

$$\mathrm{I} = (\mathrm{A}+\mathrm{a})l^2 + (\mathrm{B}+\mathrm{b})m^2 + (\mathrm{C}+\mathrm{c})n^2 - 2\mathrm{D}mn - 2\mathrm{E}nl - 2\mathrm{F}lm.$$

But it is also equal to

$$\mathrm{A}'(l+\delta l)^2 + \mathrm{B}'(m+\delta m)^2 + \mathrm{C}'(n+\delta n)^2,$$

and by substituting for $\delta l, \delta m, \delta n$, this is equal to

$$\mathrm{A}'l^2 + \mathrm{B}'m^2 + \mathrm{C}'n^2 - 2mn(\mathrm{C}'-\mathrm{B}')\alpha - 2ln(\mathrm{A}'-\mathrm{C}')\beta - 2lm(\mathrm{B}'-\mathrm{A}')\gamma$$

to the first order of small quantities.

Now this expression must be identical with the former for all values of l, m, n; hence putting $l=1$, $m=n=0$, $A'=A+a$, and similarly $B'=B+b$, $C'=C+c$. Wherefore also

$$\alpha=\frac{D}{C'-B'}=\frac{D}{C-B}\text{ nearly,}$$

and

$$\beta=\frac{E}{A-C},\ \gamma=\frac{F}{B-A};$$

and these are the required expressions for α, β, γ.

If, however, $B=A$, γ becomes infinite, and the solution is nugatory: but since, under this condition, all axes in the plane of xy were originally principal axes, the axes of reference may always be so chosen that F is zero absolutely; and then

$$\alpha=\frac{D}{C-A},\ \beta=-\frac{E}{C-A},\ \gamma=0.$$

Therefore the new principal axis C′ is inclined to the old C at a small angle $\frac{\sqrt{D^2+E^2}}{C-A}$, and is displaced along the meridian, whose longitude, measured from the plane of xz, is $\pi+\text{arc}\tan\frac{D}{E}$. This is the case to be dealt with in the present problem. The positions of the other principal axes will be of no interest.

12. *Application of preceding Theorem.*

To solve the problem numerically in any particular case, it will be necessary to find the integrals

$$D=h\varrho c^4\iint F(\theta,\phi)\sin^2\theta\cos\theta\sin\phi\,d\theta\,d\phi,$$

$$E=h\varrho c^4\iint F(\theta,\phi)\sin^2\theta\cos\theta\cos\phi\,d\theta\cdot d\phi.$$

If $\frac{D}{h\varrho c^4}$ and $\frac{E}{h\varrho c^4}$ be called d and e, then d and e stand for the above integrals, which depend on the distribution of surface-matter in continents and seas.

It will be convenient to use a foot as the unit for measuring h, and seconds of arc for the measurement of the inclination i of the new principal axis to the old. For this purpose the value of the coefficient $\frac{\varrho c^4}{C-A}$ may be calculated once for all. Let its value when multiplied by the appropriate factors for the use of the above units be called K *. Now

$$C-A=\tfrac{2}{3}\left(\varepsilon-\frac{m}{2}\right)Ma^2=\tfrac{2}{3}(1+\tfrac{2}{3}\varepsilon)\left(\varepsilon-\frac{m}{2}\right)Mc^2.$$

* I have to thank Prof. J. C. ADAMS for his help with respect to the numerical data, and for having discussed several other points with me.

Then if we take $\varepsilon = \cdot 0033439$, being the mean of the values given by Colonel A. R. CLARKE, $m = \frac{1}{289 \cdot 66}$, and $c = 20,899,917$ feet*, $M = \frac{4}{3}\pi \mathfrak{D} c^3$, and $\frac{\mathfrak{D}}{\varrho} = 2$, we get

$$C - A = \tfrac{8}{3}\pi \varrho c^4 \times \cdot 0010809 \times 20,899,917$$

and

$$K = 1 \cdot 08986.$$

If, in accordance with THOMSON and TAIT, $\frac{\mathfrak{D}}{\varrho} = 2 \cdot 1$, $K = 1 \cdot 0380$, but I shall take K as $1 \cdot 090$. Then we have $i'' = Kh\sqrt{\mathrm{d}^2 + \mathrm{e}^2}$, where $K = 1 \cdot 090$, h being measured in feet, and i'' being the angular change in the position of the principal axis of greatest moment of inertia of the earth, due to a deformation given by $hF(\theta, \phi)$ all over the surface of the spheroid.

The angle $\frac{\mathrm{a} + \mathrm{b} - 2\mathrm{c}}{C - A} . t$ is clearly of the same order of magnitude as i, as it was assumed to be in Part I.

III. FORMS OF CONTINENTS AND SEAS WHICH PRODUCE THE MAXIMUM DEFLECTION OF THE POLAR AXIS.

13. *Conditions under which the Problem is treated.*

On the hypothesis of incompressibility, the effect of a deformation in deflecting the pole is exactly equivalent to the removal of a given quantity of matter from one part of the earth's surface to another. But as no continent exceeds a few thousand feet in average height, the removal is restricted by the condition that the hollows excavated, and the continents formed, shall nowhere exceed a certain depth and height. The areas of present continents and seas, and their heights and depths, give some idea of the amount of matter at disposal, as will be shown hereafter. It is interesting, therefore, to determine what is the greatest possible deflection of the pole which can be caused by the removal of given quantities of matter from one part of the earth to another, subject to the above condition as to height and depth.

14. *Problem in Maxima and Minima.*

This involves the following problem:—To remove a given quantity of matter from one part of a sphere to another, the layers excavated or piled up not being greater than k in thickness, so as to make $\sqrt{D^2 + E^2}$ a maximum, the axes being so chosen as to make $F = 0$.

If D', E' be the products of inertia referred to other axes having the same origin and axis of z as before, it may easily be shown, from the fact that $D^2 + E^2 = D'^2 + E'^2$, that $D^2 + E^2$ is greatest and equal to E'^2 for that distribution of matter which makes $D' = 0$ and E' a maximum.

* See THOMSON and TAIT, Nat. Phil. pp. 648, 651.

The problem is thus reduced to the following:—Rectangular axes are drawn at the centre of a sphere of radius c; it is required to effect the above-described removal of matter, so that the product of inertia about a pair of planes through z, and inclined to xz at 45° on either side, shall be a maximum, subject to the above condition as to depth, k being small compared to c. For convenience, I refer to the plane xy as the equator, to xz as prime meridian, from which longitudes ψ are measured from x towards y, and to θ the colatitude. These must not be confused with the terrestrial equator, longitude, and latitude.

A little consideration shows that the seas and continents must be of uniform depth k, that there must be two of each, that they must all be of the same shape, must be symmetrical with respect to the equator, and that the continents must be symmetrical with respect to the prime meridian, and the seas with respect to meridians 90° and 270°.

Also the total product of inertia P, produced by this distribution, is 16 times that produced by the part of one continent lying in the positive octant of space; and the mass of matter removed is 8 times the mass of this same portion of one continent.

The problem is, therefore, to find the outline of the continent, so that P may be a maximum, subject to the condition that the mass is given.

Take the surface-density of the sphere as unity, and let the mass removed be given as an elevation of a height k over a fraction q of the whole sphere's surface; so that the mass removed from hollow to continent is $4\pi c^2 kq$. Then it may easily be shown that

$$\mathrm{P}=4kc^4\int_0^{\frac{\pi}{2}}\sin^3\theta\sin 2\psi d\theta$$

and

$$q=\frac{1}{\pi}\int_0^{\frac{\pi}{2}}2\psi\sin\theta d\theta,$$

where ψ is a function of θ to be determined. Then writing ω for 2ψ, and μ for $\cos\theta$, we have to make

$$\int_0^1\{(1-\mu^2)\sin\omega-\omega\cos^2\alpha\}d\mu \text{ a maximum,}$$

for it will be seen later that $-c^2\cos^2\alpha$ is a proper form for the constant, to be introduced according to the principles of the Calculus of Variations. This leads at once to

$$(1-\mu^2)\cos\omega=\cos^2\alpha$$

or

$$\sin^2\theta\cos 2\psi=\cos^2\alpha,$$

That is to say, the outline of the continent is the sphero-conic formed by the intersection with the sphere of the cone, whose cartesian equation is

$$y^2(1+\cos^2\alpha)+z^2\cos^2\alpha=x^2\sin^2\alpha.$$

Reverting to the expressions for P and q, altering the variable of integration, and the limits, so as to exclude the imaginary parts of the integrals, we have as the equation to find α

$$q=\frac{1}{\pi}\int_0^\alpha \cos\chi \operatorname{arc}\cos\left(\frac{\cos\alpha}{\cos\chi}\right)^2 d\chi,$$

and

$$P=4kc^4\int_0^\alpha \cos\chi\sqrt{\cos^4\chi-\cos^4\alpha}\,d\chi.$$

These integrals are reducible to elliptic functions; but in order not to interrupt the argument, I give the reduction in Appendix B. If $\cos 2\gamma=\cos^2\alpha$, the result is that

$$\pi q=\sqrt{2}\,\frac{\cos 2\gamma}{\cos\gamma}\left[\Pi^1(-2\sin^2\gamma)-F^1\right]$$

or

$$q=1-\frac{2}{\pi}\{E^1F-F^1(F-E)\}$$

and

$$\frac{P}{kc^4}=\frac{8\sqrt{2}}{3}\cos\gamma\left[E^1-\cos 2\gamma F^1\right],$$

where the modulus of the complete functions E^1, F^1, Π^1 is $\tan\gamma$, and where E, F have a modulus $\frac{\cos\alpha}{\cos\gamma}$ and an amplitude $\frac{\pi}{2}-\gamma$.

It will be observed that α is the semi-length of the continent in latitude, and γ the semi-breadth in longitude.

From these expressions I have constructed the following Table:—

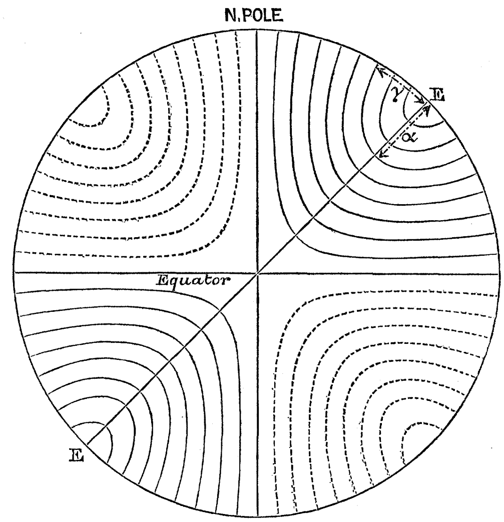
Semi-breadth of continent (γ).	Semi-length of continent (α).	Fraction of surface elevated or depressed (q).	Product of inertia $\left(\frac{P}{kc^4}\right)$.
0°	0°	·0000	·0000
5	7 5	·0054	·0672
10	14 13	·0216	·2628
15	21 28	·0486	·5697
20	28 55	·0867	·9603
25	36 42	·1362	1·3981
30	45 0	·1979	1·8399
35	54 12	·2732	2·2371
40	65 22	......	......
45	90 0	·5000	2·6667

15. *Application of preceding problem to the case of the Earth.*

In the application to the case of the earth, what has been called, for brevity, the equator (EE in fig. 2) must be taken as a great circle, passing through a point in *terrestrial* latitude 45°.

Figure 2 gives the stereographic projection of the forms of continents and seas the firm lines showing continents, and the broken ones seas, when covering various fractions of the whole surface; α and γ are indicated on one of the continents. The other hemisphere is the same as this figure, when seen in a looking-glass. It will be observed that the limiting case is when the two continents fill up two quarters of the earth, and the two seas the other two.

Fig. 2.



It is clear that the greatest deflection of the polar axis which can be produced by the elevation of continents of height k and having a total area $4\pi c^2 q$, and the depression of similar seas, will be PK.

A numerical Table of results will be given below, formed by interpolation in the above Table.

IV. ON GEOLOGICAL CHANGES ON THE EARTH'S SURFACE.

16. *The points to be considered.*

It is now necessary to consider what kind and amount of superficial changes are brought about in the earth's shape by such geological changes as are believed to have taken place. The points to be determined are:—

i. Over what extent of the earth's surface is there evidence of consentaneous subsidence, or upheaval, during any one period.

ii. What is the extreme vertical amount of that subsidence or upheaval.

iii. How the sea affects the local excesses and deficiencies of matter on the earth's surface.

iv. How marine and aërial erosion affect the distribution of the excess or deficiency of matter.

v. The possibility of wide-spread deformations of the earth, which approximately carry the level surfaces with them.

The object of this discussion is to find what areas and amounts of elevation and subsidence on a sealess and rainless globe are equivalent, as far as moving the principal axis, to those which obtain on the earth. These areas and effects will be referred to as "effective areas and amounts of elevation or subsidence."

It is probable that during the elevation or subsidence of any large area, the change proceeds at unequal rates in different parts; probably one part falls or rises more quickly than another, and then the latter gains on the former. But it has been shown, in the dynamical part of this paper, that the axis of rotation sensibly follows the axis of figure. Hence it is immaterial by what course the earth changes its configuration, provided the changes do not proceed by large impulses, a supposition which may be certainly excluded.

The essential point is, to compare the final and initial distributions of matter, after and before a period of large geographical change.

17. *Areas of subsidence and elevation.*

When a new continent is being raised above the sea, there is no certainty as to the extent to which areas in the adjoining seas partake in the elevation; even in the case of S. America, where the area of elevation is supposed to be abruptedly limited towards the west, the line of 15,000 feet depth lies a long way from the coast.

As soon, moreover, as the land is raised above the sea, the rivers begin washing away its surface, and the sea eating into its coasts. The materials of the land are carried away, and deposited in the surrounding seas. Thus to form a continent of 1000 feet in height, perhaps entails an elevation of the surface of from 3000 to 4000 feet, and all the matter of the additional 2000 to 3000 feet is deposited in the sea. This tends to make the adjoining seas shallower, and to cause some increase to the area of the land. Therefore in a sealess globe the effect must be represented by a greater area of elevation and a less height.

The bed of a deep sea is hardly at all subject to erosion, and therefore the tendency seems to be to make the negative features of an ocean-bottom more pronounced than the positive features of mountain-ranges, at least in the parts very remote from land.

The areas, then, of existing continents may not be a due measure of the areas of effective elevation; we can only say that the latter may considerably exceed the former. The direct evidence as to the extent of the earth's surface over which there has been a general movement during any one period, is also very meagre. It appears certain that very large portions of S. America have undergone a general upward movement within a recent geological period; but there is no certainty whatever as to the limits of this area, nor as to whether the beds of the adjoining seas have partaken to any extent of this general movement. Thus the case of S. America is of scarcely any avail in determining the point in question. The presence of deep ocean up to the Chilian coast seems, however, to make it probable that areas of elevation are more or less abruptly divided from those of rest or subsidence.

There is only one area of large extent in which we possess fairly well-marked evidence of a general subsidence; and this is the area embracing the Coral islands of the Pacific Ocean. The evidence is derived from the structure of the Coral islands, and is confirmed in certain points by the geographical distribution of plants and animals. Some naturalists are of opinion that there is evidence of the existence of a previous continent; others (and amongst them my father, Mr. CHARLES DARWIN) that there existed there an archipelago of islands. In this dearth of precise information, only a rough estimate of area is possible.

My father, who has especially attended to the subject of the subsidence of the Pacific islands, has marked for me, on the map given in his work on Coral Reefs, a large area which he believes to have undergone a general subsiding motion. This area runs in a great band from the Low Archipelago to the Caroline Islands, and embraces the greater number of the islands coloured dark-blue in his map. The boundary may be defined as passing through:—

	N.							S.							
Lat.	3	5	15	22	18	10	5	5	15	25	30	18	15	10	8
Long......	150	140	150	165	180	165	150	135	120	120	135	150	165	180	165
	E.					W.									E.

He also marked a smaller area, embracing New Caledonia, the S.E. corner of New Guinea, and the N.E. coast of Australia.

It is noteworthy that the former large area consists of sea more than 15,000 feet deep, except in patches round some of the islands, where it appears to be from 10,000 to 15,000 feet deep*.

I marked these areas on a globe, and cut out a number of pieces of paper to fit them, and then weighed them. By this method I determined that the former area was ·055 of the whole surface of the globe, and the latter was ·01; the two together were therefore ·065.

It thus appears that we have some evidence of an area of between 5 and 7 per cent. of the globe having undergone a general motion of subsidence within a late geological period. But between this area and the coast of S. America there is a vast and deep ocean, and nothing whatever is known with respect to the movements of its bed. Hence it is quite possible that the area which has really sunk, in this quarter of the globe, is considerably larger than the one above spoken of.

On the whole, then, perhaps from ·05 to ·1 of the whole surface may at various times have partaken of a consentaneous movement, so as to convert deep sea into land, and *vice versâ*.

Besides this kind of general movement, there have certainly been many more or less local rises and falls, but this small oscillation is not fitted to produce any sensible effect on the position of the earth's axis.

18. *Amount of Elevation, and the effects of Water.*

HUMBOLDT has shown that the mean height of the present continents is a little less than 1100 feet from the sea-level†. But this, of course, does not give the limit to the amount of change of level. On the other hand, there are perhaps 50,000 to 80,000 feet of superposed strata at most places on the earth; but neither does this give the indication required, because the surface must have risen and fallen many times during the deposition of these strata.

But, as before pointed out, the actual upward or downward movement of land is by

* See frontispiece-map to WALLACE's Geogr. Distrib. of Animals.

† Sir J. HERSCHEL seems to have doubled the height through a misconception of HUMBOLDT's meaning. The mean height of the land is in English feet: Europe, 671; N. America, 748; Asia, 1132; S. America, 1151. See a letter to 'Nature,' by Mr. J. CARRICK MOORE, April 18th, 1872.

no means the same as its effective elevation or subsidence; for erosion causes the effective to be far slower than the actual. And the actual upward or downward movement of an ocean-bed is different from the effective; for the sea-water will flow off or in from the adjoining seas. The specific gravity of water is about one third of that of surface rock, and the local loss or gain of matter is the actual loss or gain of surface rock, less the mass of the sea-water admitted or displaced. Thus the effective downward or upward movement of a sea-bed is about $\frac{2}{3}$ of the actual; of this a more accurate estimate will be given presently.

It is fortunately not important to track the series of changes through their course; and in order to avoid the complication of doing so, the way seems to be to estimate the amount of transference of matter entailed in the conversion of a deep ocean into a continent of the present mean height.

Suppose, then, that an ocean area of 15,000 feet in depth were gradually elevated, and that the final result, notwithstanding erosion, were a continent of 1100 feet in height. Conceive a prism, the area of whose section is unity, running vertically upwards from what was initially the ocean-bed. Initially this prism contained 15,000 feet of sea-water, and finally it contains 16,100 feet of rock; so that the local gain of matter, on this unit of area of the earth's surface, is the difference between the masses of this prism, initially and finally.

Now 1·02 is the specific gravity of sea-water, and 2·75 that of surface rock; therefore the same local gain of matter, in a sealess globe, would be given by an elevation of

$$16{,}100-\tfrac{102}{275}\text{ of }15{,}000=10{,}436\text{ feet.}$$

That is to say, 10,436 feet has been the effective elevation.

I therefore adopt 10,000 feet as the effective elevation equivalent to the conversion of deep ocean into a continent; and in the examples given hereafter, where I find the deflection of the pole for various forms and sizes of continent, I shall give the results of such an assumed conversion.

19. *Wide-spread Deformations of the Earth.*

It has hitherto been assumed that the elevation of land would not affect the sea-level; but there can be no doubt but that elevations, such as those already spoken of, would do so to the extent of, say, a hundred feet. In so far, then, as this is the case, the elevation would be masked from the eyes of geologists. But if the change of form were a gradual rising over a very wide area, the level surfaces would approximately follow the form of the rocky surface. For instance, the elliptical form of the equator carries the ocean level with it; the amount of this ellipticity is such that the difference between the longest and shortest equatorial radii is 6378 feet*. So long, however, as these bulges remain equatorial they cannot affect the position of the principal axis, even should they vary in amount from time to time. But this kind of deformation, if not symmetrical

* THOMSON and TAIT, Nat. Phil. p. 648.

with respect to the equator, would alter the position of the principal axis, without leaving any trace whatever of elevation or depression for geologists to discover.

The discrepancy which is found between the ellipticity of the earth, as deduced from various arcs of meridian, is, I presume, attributable to real inequalities in the earth's form, and not entirely to errors of observation and to the elliptical form of the equatorial section. It seems, moreover, quite possible that these wide-spread inequalities may have varied from time to time.

Hence, even if the deposit of strata in the sea did not produce a continual shifting of the weights on the earth's surface, and even if geologists should ultimately come to the conclusion that there has never been any consentaneous elevation and depression of very large continents relative to the sea-level, but that the oscillations of level have always been local, it would by no means follow that the earth's axis has remained geographically fixed.

V. NUMERICAL APPLICATION TO THE CASE OF THE EARTH.

20. *Continents and Seas of Maximum Effect.*

As far as I can learn, geologists are not of opinion that there is any more reason why upheavals and subsidences should take place at one part of the earth's surface than at another. It is accordingly of interest to suppose the elevations and depressions to take place in the most favourable places for shifting the axis of figure. The area over which a consentaneous change may take place is also a matter of opinion.

The theorem in maxima and minima in Part III. makes it easy to construct a table from which that area may be selected which seems most probable to geologists. The following Table is formed by interpolation in the Table in sec. 14; the first column gives the fraction of the earth's surface over which an elevation is supposed to take place, a depression over an equal area taking place simultaneously. The second column gives the angular shift in the earth's axis of figure, due to 10,000 feet of effective elevation; as was shown in Part IV., this would convert a deep ocean into a continent. If 10,000 feet be thought too high an estimate, the last column may be reduced in any desired proportion. Lastly, fig. 2 shows the forms of these continents and seas of maximum effect.

Area of elevation *or* subsidence, as fraction of Earth's surface.	Deflection of pole for 10,000 feet effective elevation.
·001	$2\frac{1}{4}'$
·005	$11\frac{1}{3}'$
·01	$22\frac{1}{2}'$
·05	$1^\circ\ 46\frac{1}{2}'$
·1	$3^\circ\ 17'$
·15	$4^\circ\ 33\frac{2}{3}'$
·2	$5^\circ\ 36\frac{2}{3}'$
·5	$8^\circ\ 4\frac{1}{2}'$

N.B. The area of Africa is about ·059, and of S. America about ·033 of the Earth's surface.

21. *Examples of other forms of Continent.*

I will now apply the preceding work to a few cases where the continents and seas do not satisfy the condition of giving the maximum effect.

Figures 3, 4, 5, and 6 represent the shapes of the continents as projected stereographically. The shaded parts represent areas of elevation, the dotted parts those of depression; and in the shelving continents and seas the contour lines are roughly indicated. P′ shows the new position of the pole. In every case here given $d=0$ and $F=0$.

Fig. 3. $F(\theta, \phi)=\sin 2\theta \cos 2\phi$, from $\theta=0$ to π, and $\phi=-\frac{\pi}{4}$ to $+\frac{\pi}{4}$, and zero over the rest of the globe.

$$e=2\int_0^{\pi}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^3\theta \cos^2\theta \cos 2\phi \cos\phi \, d\theta \, d\phi=\tfrac{16}{45}\sqrt{2},$$

$$i''=Khe=\cdot 5480h.$$

If the effective elevation or depth in the middle of continent or sea be 10,000 feet, $PP'=1^\circ\ 31\tfrac{1}{3}'$.

This is the form of continent for which $\mathfrak{H}_2$ is worked out in Appendix A.

Fig. 4. The same shape as the last, but of uniform elevation and depression of 10,000 feet.

$$e=2\int_0^{\frac{\pi}{2}}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2\theta \cos\theta \cos\phi \, d\theta \, d\phi=\tfrac{2}{3}\sqrt{2},$$

$$i''=Khe=1\cdot 028\times h.$$

$PP'=2^\circ\ 51\tfrac{1}{4}'$ when $h=10{,}000$; an extreme supposition, as the area affected is a quarter of the whole globe.

Fig. 5. $F(\theta, \phi)=1$, from $\theta=0$ to $\frac{\pi}{2}$, and from $\phi=-\frac{\pi}{4}$ to $+\frac{\pi}{4}$, and $-\frac{1}{7}$ over the rest of the globe. This is equivalent to $F(\theta, \phi)=\frac{8}{7}$ within the above limits.

Then $$i''=\tfrac{4}{7}\times 1\cdot 028\times h=\cdot 587\times h,$$

and $$PP'=1^\circ\ 38', \text{ when } h=10{,}000 \text{ feet}.$$

Fig. 6. $F(\theta, \phi)=\sin 2\theta \cos 2\phi$, from $\theta=0$ to $\frac{\pi}{2}$, and from $\phi=-\frac{\pi}{2}$ to $+\frac{\pi}{2}$, and zero over the rest of the globe.

$$e=2\int_0^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^3\theta \cos^2\theta \cos 2\phi \cos\phi \, d\theta \, d\phi=\tfrac{8}{45},$$

$$i''=Khe=\cdot 194\times h;\ PP'=32\tfrac{1}{3}', \text{ when } h=10{,}000 \text{ feet}.$$

On the whole, then, it appears that continents, such as those with which we have to deal, are competent to produce a geographical alteration in the position of the pole of between one and three degrees of latitude. But all these results are obtained on what I have called the hypothesis of incompressibility.

Fig. 3.

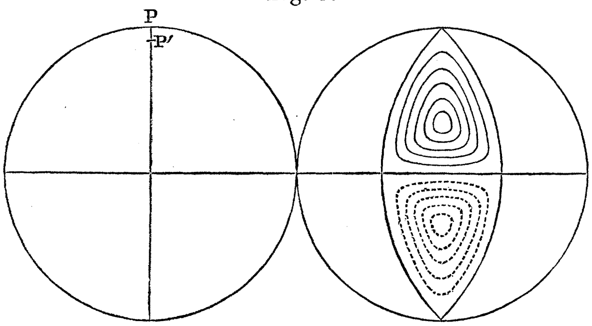


Fig. 4.

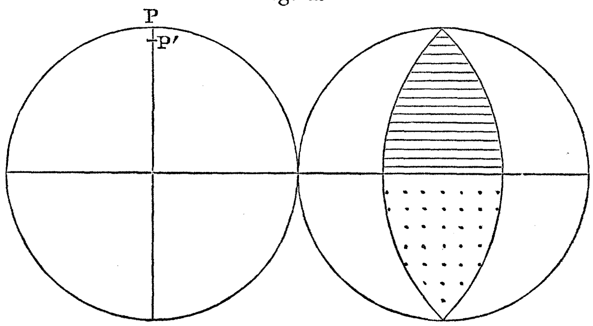


Fig. 5.

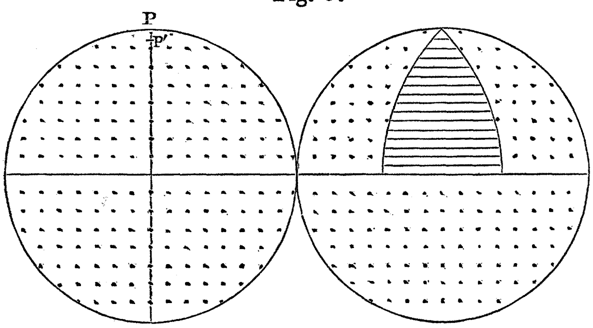
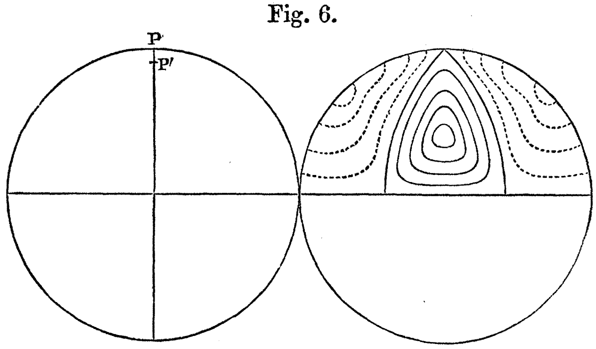


Fig. 6.



VI. HYPOTHESES OF INTERNAL CHANGES OF DENSITY ACCOMPANYING ELEVATION AND SUBSIDENCE.

22. *A general Shrinking of the Earth.*

It may be supposed that the earth is gradually shrinking, but that it shrinks quicker than the mean in some regions and slower in others. This would of course lead to depression and elevation above and below the mean surface in those regions. A deformation of this kind may be represented as a uniform compression of the earth, superposed on changes such as those considered on the hypothesis of incompressibility. If α be the coefficient of contraction of volume, it is clear that the values of D and E, as already found, must be diminished in the proportion of $1-\frac{2\alpha}{3}$ to unity, and C$-$A must be diminished in the like proportion. Hence the deflections of the polar axis, on this hypothesis, are exactly the same as those already found. This seems, perhaps, the most probable theory, but it is well to consider others.

The redistribution of matter caused by the erosion of continents will clearly produce the same effect as deformations on the theory of incompressibility.

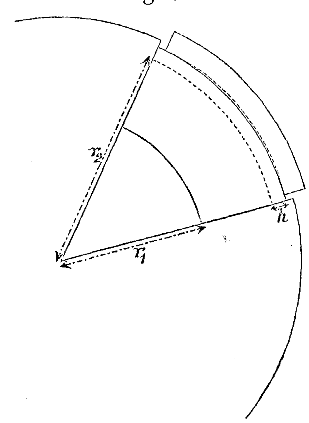
23. *Changes of Internal Density producing Elevation.*

In discussing the above hypothesis, I shall confine myself to the case of the upheaval or subsidence being of uniform height over given areas, and shall make certain other special assumptions. This will considerably facilitate the analysis, and will give sufficient insight into the extent to which previous results will be modified.

I assume, then, that the elevation of the surface is produced by a swelling of the strata contained between distances r_1 and r_2 from the centre of the globe and immediately under the area of elevation, and that the coefficient of cubical expansion α is constant throughout the intumescent portion.

This will cause a fracture of the strata of equal density, and will produce a discontinuity such as that shown in figure 7, where the dotted circle of radius r_2 indicates the upper boundary of the swelling strata before their intumescence.

Fig. 7.



But the shift of the earth's axis, caused by this kind of deformation, will differ insensibly from what would obtain if there were a more or less abrupt flexure of the strata of equal density at the boundaries of the intumescent volume and of the area of elevation.

Suppose, as before, that h is the height to which the continent is raised above the surface; then we require to know α in terms of h.

Before intumescence, let r, θ, ϕ be the coordinates of any point within the intumescent volume; and suppose that r becomes $r+u$, whilst θ and ϕ, of course, remain constant.

The equation of continuity is easily found to be

$$\frac{du}{dr}+\frac{2u}{r}=\alpha,$$

of which the integral is $ur^2=\frac{\alpha}{3}r^3+\beta$.

If β be determined, so that when $r=r_1$, $u=0$,

$$u=\frac{\alpha}{3}\frac{r^3-r_1{}^3}{r^2}.$$

But when $r=r_2$, $u=h$, the elevation of the surface; therefore

$$\alpha=\frac{3h}{r_2}\frac{1}{1-\left(\frac{r_1}{r_2}\right)^3},$$

the required expression for α in terms of h.

Also, before intumescence, LAPLACE's law of internal density held good, viz. $Q\frac{\sin qr}{r}$, therefore afterwards the density of the stratum distant $r+u$ from the centre is $Q(1-\alpha)\frac{\sin qr}{r}$.

Now the propositions given in Part II., as to the change in the position of the earth's axis, remain true here also; and the only difference is that the products of inertia D and E must now be expressed by different integrals.

After intumescence the earth may be conceived to consist of:—*first*, itself as it was before; *secondly*, of *negative* matter, of which the law of density is $Q\frac{\sin qr}{r}$, throughout the space bounded by $r=r_1$, $r=c$, and the cone of elevation; and, *thirdly*, of the matter which formerly lay within this space, in the configuration attained by it after intumescence.

The first part clearly contributes nothing to D and E; and the second contributes

$$-Q\iiint r^3 \sin qr \sin^2\theta \cos\theta \begin{Bmatrix}\sin\phi\\ \cos\phi\end{Bmatrix} dr\,d\theta\,d\phi,$$

integrated throughout the above space, that is from $r=r_1$ to $r=c$, and throughout the cone of elevation.

As to the third part, the mass of any element remains unchanged, whilst its distance from the centre has become $r+u$. Hence the third part contributes

$$Q\iiint r(r+u)^2 \sin qr \sin^2\theta \cos\theta \begin{Bmatrix}\sin\phi\\ \cos\phi\end{Bmatrix} dr\,d\theta\,d\phi,$$

integrated throughout the above space.

Therefore, taking all together, and treating u as small,

$$\left.\begin{matrix}D\\E\end{matrix}\right\} = 2Q\iiint ur^2 \sin qr \sin^2\theta \cos\theta \begin{Bmatrix}\sin\phi\\ \cos\phi\end{Bmatrix} dr\,d\theta\,d\phi.$$

Therefore

$$\frac{D}{d}=\frac{E}{e}=2Q\int ur^2 \sin qr\,dr,$$

where d and e have the same meanings as before, in Part II. sec. 12.

Now this last integral divides itself into two parts: first, from $r=c$ to $r=r_2$, $u=h$; and, secondly, from $r=r_2$ to $r=r_1$, $u=\frac{\alpha}{3}\cdot\frac{r^3-r_1^3}{r^2}$.

Therefore

$$\frac{D}{d}=\frac{E}{e}=2Qh\int_{r_2}^{c} r^2 \sin qr\,dr+\tfrac{2}{3}Q\alpha\int_{r_1}^{r_2}(r^3-r_1^3)\sin qr\,dr.$$

If the value of α be substituted, and the integrations effected, it will be found that

$$\frac{D}{2dQhc^3}=\frac{E}{2eQhc^3}=-\frac{1}{qc\cos qc}+\frac{2}{(qc)^2}\frac{S}{\sin qc}+\frac{3c}{r_2}\cdot\frac{1}{(qc)^2}\frac{1}{1-\left(\frac{r_1}{r_2}\right)^3}\frac{T}{\sin qc}=U \text{ suppose,}$$

where S stands for the expression $\frac{r}{c}\sin qr+\cos qr$, taken between the limits c and r_2, and T for the expression $\left(\frac{r}{c}\right)^2\sin qr+2\frac{r}{c}\cdot\frac{1}{qc}\cos qr-\frac{2}{(qc)^2}\sin qr$, taken between the limits r_2 and r_1.

Substituting in the expression $i=\frac{\sqrt{D^2+E^2}}{C-A}$, and using the coefficient K, we get

$$i''=2KUh\sqrt{d^2+e^2}.$$

It must be noticed that this investigation is applicable as much to subsidence caused by internal compression as it is to elevation; and the word intumescence is used to cover both phenomena. In the case of subsidence h is negative.

Now on the hypothesis of incompressibility it was shown that $i''=Kh\sqrt{d^2+e^2}$. Hence, on the present hypotheses, the estimated deflection of the pole must be diminished in the proportion of 2U : 1.

Taking $qc=141°$ (which makes $\frac{\mathfrak{D}}{\varrho}=2$, very nearly), I have calculated the values of 2U, when $\frac{r_1}{c}=\frac{399}{400}$, and $\frac{r_2}{c}=\frac{79}{80}, \frac{9}{10}, \frac{3}{4}, \frac{1}{2}, 0$. If the earth's radius be taken as 4000 miles, this gives, that the superficial strata for 10 miles in thickness do not swell, but are merely heaved up, and that the lower surface of the intumescent volume is at the various distances from the earth's surface given in the first column of the following Table. The second column gives 2U, or the factor by which previous results would have to be diminished on the present hypothesis. The third column gives the so diminished value of 1° of deflection of the pole.

Depth below surface of bottom of intumescent volume, in miles, $(c-r_2)$.	Factor of diminution of former results, (2U).	A deflection of 1° would be reduced to (2U×1°).
50	·0126	46″
400	·1011	6′ 4″
1000	·2731	16′ 23″
2000	·5171	31′ 2″
4000	·6721	40′ 20″

The last row, of course, indicates that the intumescence extends quite down to the centre of the earth.

This Table shows that if elevation is due to the swelling of strata at all near the surface, the alteration in the position of the polar axis would be reduced to quite an insignificant amount. The alleged deficiency of density under the Himalayas affords some slight evidence that it is so, at least occasionally. I believe, also, that Mr. MALLET is of opinion that the centre of disturbance of earthquake-shocks is not at a greater distance than 30 miles below the surface*. It does not, of course, follow from this evidence that there may not be elevations of both kinds going on, some being approximately superficial phenomena, and others probably due to unequal shrinking of the earth as a whole. The latter kind would be likely to produce more extensive deviations from the external form of equilibrium than the former.

* Referred to at second hand by Mr. CARRUTHERS, Trans. New-Zeal. Inst. vol. viii. p. 363.

On the whole, then, it appears that the deflection of the polar axis cannot exceed that which was found in the case of incompressibility, and it may possibly be considerably less. The complete want of knowledge of the internal movements only allows us to state a superior limit to the change which might be produced by any one upheaval or subsidence.

VII. SUMMARY AND CONCLUSION.

24. *Summary.*

For the sake of those who do not read mathematics, I will shortly recapitulate the chief results arrived at.

The change in the obliquity of the ecliptic caused by any gradual deformation of the earth's shape of small amount is very small. Even so great a redistribution of weights on the earth's surface as is entailed by immense polar ice-caps during the Glacial Period, cannot have altered the obliquity by so much as $\frac{1}{2200}$ of a second of arc; and this is the most favourable redistribution of weights for producing this effect. Thus throughout geological history the obliquity of the ecliptic must have remained sensibly constant. And, further, when the earth undergoes any such deformation, the axis of rotation follows, and remains sensibly coincident with the principal axis of figure.

It thus only remains to consider the change in the geographical position of the poles caused by the deformation.

The principal axes at the centre of inertia of a body are three lines mutually perpendicular, and their position is entirely determined by the shape of the body. Hence if a nearly spherical body be slightly deformed, the extremities of these principal axes will move from their original positions and describe paths on the surface of the body, which may be shortly described as the paths of the principal axes. In the case of the earth, as geologically deformed, it is only of interest to consider the path of one of these axes, which is, in common parlance, the earth's axis.

If the earth be sensibly rigid, or should only readjust itself to an approximate form of equilibrium at long intervals (as maintained in Part I.), the geographical path of the axis is very nearly the same as is due merely to the geological deformation of the earth's shape; but if the earth be more or less plastic, or should readjust itself frequently to an approximate form of equilibrium, the dynamical reactions introduced are such as more or less to modify the geographical path of the axis. In the case of great plasticity these reactions would suffice to entirely alter the character of the path. It seems probable that during the consolidation of the earth there was great instability in the geographical position of the poles. Throughout the rest of the investigation suppositions of plasticity are set aside, and the hypothesis of sensible rigidity is adhered to.

Formulæ for the change in the geographical position of the pole due to any small deformation are found in Part II.

On the assumption that the internal density of the earth remains unchanged by the deformation, the forms of continent and depression which produce the greatest deflection of the poles, for the transport of a given quantity of matter from one part of the earth's surface to another, are then investigated. These forms are shown, projected stereographically, in fig. 2 (p. 293).

Part IV. gives what evidence I have been able to collect of the areas and amounts of deformation to which the earth may have been subjected in geological history; but as the discussion is not mathematical, it seems unnecessary to give an abstract thereof.

Part V. gives numerical applications of the preceding theorems to the case of the earth, on the assumption that the internal density is unaltered by the deformation. From this it appears that the poles may have been deflected from 1° to 3° in *any one geological period*; but the reader is referred back to that part for details.

If upheaval and subsidence of the surface are due to a shrinking of the earth as a whole, but to a more rapid shrinking in some regions than others, the deflection of the poles is the same as that found where there is no disturbance of the strata of equal density.

But if the upheaval and subsidence are due to local intumescence and contraction of the strata underneath the rising or falling areas, the previous numerical estimates must be largely reduced; for the extent of this reduction the reader is referred to the Table in Section 23 (p. 302).

It thus appears that the deflection of the poles first given is a superior limit to that which is possible.

25. *Conclusion.*

There remain, in conclusion, one or two miscellaneous points to be referred to.

In a letter to Sir C. LYELL read before the Geological Society*, Sir JOHN HERSCHEL has pointed out that the isothermal strata near the surface of the earth must approximately follow the solid surface. Therefore, when a thick stratum is deposited at the bottom of the ocean, the primitive bottom is gradually warmed and expands. There is thus a tendency for the upheaval of sea-beds, on which a large amount of matter has been deposited; but this kind of upheaval certainly falls within the case of superficial intumescence, and could therefore affect the geographical position of the poles but little more than would be due merely to the weight of the deposited stratum. It must be noticed, moreover, that the weight of the deposited stratum would tend to compress the primitive sea-bed, and might counteract the expansion due to rise of temperature.

If the earth were absolutely rigid the pole could never have wandered more than from 1° to 3° from its primitive position, whatever geological changes were successively to take place; because the new pole could never be brought to a greater distance from its original position, by any fresh distribution of the matter forming the continents, than the maximum for this amount of matter arranged in continents of a like height.

But it was maintained in Part I. that from time to time the earth makes a kind of

* Proc. Geol. Soc. vol. ii. p. 549.

rough adjustment to a figure of equilibrium. If this adjustment is, as seems probable, by an earthquake, it will take place with reference to the axis of rotation at the instant of the earthquake. Now there exists in erosion and marine deposits a cause of terrestrial deformation which is certainly independent of such adjustments; and it seems probable that the causes of geological upheaval and subsidence are so also. We have therefore clearly a state of things in which the pole may wander indefinitely from its primitive position. On this hypothesis, as in successive periods the continents have risen and fallen, the pole may have worked its way, in a devious course, some 10° or 15° away from its geographical position at consolidation, or may have made an excursion of smaller amount and have returned to near its old position. May not the Glacial Period, then, have been only apparently a period of great cold? If at that period the N. pole stood somewhere where Greenland now stands, would not the whole of Europe and a large part of N. America have been glaciated? And if the N. pole retreated to its present position, would it not leave behind it the appearance of a very cold climate having prevailed in those regions?

But although such a cumulative effect is possible with respect to the geographical position of the pole, none such is possible with respect to the obliquity of the ecliptic.

Now this kind of wandering of the poles would of course require extensive and numerous deformations, and it is hard to see how there can have been a shifting of the surface weights sufficient to produce it, without frequent changes in the geographical distribution of land and water. If, then, geologists are right in supposing that where the continents now stand they have always stood, would it not be almost necessary to give up any hypothesis which involved a very wide excursion of the poles?

APPENDIX A. (See p. 279.)

To calculate $\mathfrak{H}_1$ and $\mathfrak{H}_2$ in a supposed case of elevation and subsidence.

Take the case of sec. 21 (fig. 3), where the elevation is given by $ht \sin 2\theta \cos 2\phi$, from $\theta=0$ to π, and from $\phi=-\frac{\pi}{4}$ to $\frac{\pi}{4}$, and zero over the rest of the sphere. Suppose that the internal motion is entirely confined to the quarter of the sphere defined by the above limits of θ and ϕ, that radial particles are always radial, and that the motion is entirely meridional.

Let $\theta+\vartheta$ be the disturbed colatitude of the point θ, ϕ. Then the equation of continuity, which expresses that the volume of the elementary pyramid $\frac{1}{3}c^3 \sin\theta \, d\theta \, d\phi$ remains constant, when θ becomes $\theta+\vartheta$, is

$$\frac{d}{d\theta}(\vartheta \sin\theta)+\frac{3ht}{c}\sin\theta \sin 2\theta \cos 2\phi=0,$$

the integral of which is

$$\vartheta \sin\theta+\frac{2h}{c}t\cos 2\phi \sin^3\theta=\text{a constant};$$

and since ϑ is zero, when $\phi=\pm\frac{\pi}{4}$, for all values of t, $\vartheta=-\frac{2h}{c}t\cos 2\phi\sin^2\theta$, and

$$\frac{d\vartheta}{dt}=-\frac{2h}{c}\cos 2\phi\sin^2\theta.$$

Hence H_2, twice the area conserved on the plane of xz, is

$$\iiint \varrho r^2\sin\theta\, dr\, d\theta\, d\phi\,.\,\frac{r^2 d\vartheta}{dt}\cos\phi,$$

taken from $r=0$ to c, $\theta=0$ to π, $\phi=-\frac{\pi}{4}$ to $+\frac{\pi}{4}$.

If the sphere be taken as homogeneous,

$$H_2=-\frac{2h}{c}\varrho\iiint r^4\sin^3\theta\cos 2\phi\cos\phi\, dr\, d\theta\, d\phi$$

$$=-\frac{16}{45}\sqrt{2}\, h\varrho c^4=-\frac{4\sqrt{2}}{15\pi}\mathrm{M}hc,$$

H_1 and H_3 are both clearly zero.

The above value of H_2 is larger than what it would be in the case of the earth, if LAPLACE's law of internal density were true. because the external layers have been taken too heavy, and the internal too light. But taking that law of density, $\mathrm{A}=\frac{1}{3}\mathrm{M}c^2$ very nearly.

Hence $\frac{H_2}{\mathrm{A}}=-\frac{4\sqrt{2}}{5\pi}\frac{h}{c}$.

If we let the time run on until the highest point of the continent has risen one foot, so that $\frac{ht}{c}=\frac{1}{20{,}900{,}000}$, then $\frac{H_2 t}{\mathrm{A}}=-\frac{4\sqrt{2}}{5\pi}\frac{1}{20{,}900{,}000}$.

But reference to sec. 21 (fig. 3) shows that $i''=\cdot 5480h$, or in the present notation,

$$\beta t=\cdot 5\times\frac{\pi}{648{,}000}\text{ nearly.}$$

Therefore

$$\frac{H_2}{\mathrm{A}\beta}=-\frac{8\times 648\sqrt{2}}{104{,}500\pi^2}=-\frac{1}{141}\text{ nearly.}$$

But generally, since the angular velocities α, β, γ of the moving axes, to which $\mathfrak{H}_1$, $\mathfrak{H}_2$, $\mathfrak{H}_3$ refer, are very small, therefore

$$\mathfrak{H}_1=H_1,\ \mathfrak{H}_2=H_2,\ \mathfrak{H}_3=H_3$$

to the first order of small quantities, within the limited period to which the investigation applies. So that in this particular case,

$$\frac{\mathfrak{H}_2}{\mathrm{A}\beta}=-\frac{1}{141}\text{ nearly, and }\mathfrak{H}_1=\mathfrak{H}_3=0.$$

And, besides, this value of $\frac{\mathfrak{H}_2}{\mathrm{A}\beta}$ is larger than it ought to be, because $\mathfrak{H}_2$ was calculated on an assumed homogeneity of the earth. This, then, justifies the conclusion in the text on p. 279.

In the elevation and subsidence given by $ht \sin 2\theta \sin 2\phi$ from $\theta=0$ to $\frac{\pi}{2}$, and from $\phi=-\frac{\pi}{2}$ to $\frac{\pi}{2}$, H_1 and H_2 are clearly zero, under a like supposition as to the nature of the internal motions accompanying upheavals.

APPENDIX B. (See p. 292.)

To reduce the integrals $\int_0^\alpha \cos\chi \operatorname{arc\,cos} \frac{\cos^2\alpha}{\cos^2\chi}\, d\chi$ *and* $\int_0^\alpha \cos\chi \sqrt{\cos^4\chi - \cos^4\alpha}\, d\chi$ *to elliptic functions.*

Call the former A and the latter B.

Then integrating A by parts,

$$A=-\int_0^\alpha \sin\chi\, d\left(\operatorname{arc\,cos}\frac{\cos^2\alpha}{\cos^2\chi}\right).$$

Put $x=\sin\chi$, and $\cos 2\gamma=\cos^2\alpha$, then we get

$$A=2\cos 2\gamma\int_0^{\sqrt{2}\sin\gamma}\left(\frac{1}{1-x^2}-1\right)\frac{dx}{\sqrt{x^4-2x^2+\sin^2 2\gamma}};$$

and if $x=\sqrt{2}\sin\gamma\sin\phi$, this becomes

$$\sqrt{2}\,\frac{\cos 2\gamma}{\cos\gamma}\{\Pi^1(-2\sin^2\gamma)-\mathrm{F}^1\}, \text{ where the modulus is } \tan\gamma.$$

Again, integrating B by parts,

$$B=\int_0^\alpha \sin\chi\cdot\frac{4\cos^3\chi\sin\chi\, d\chi}{2\sqrt{\cos^4\chi-\cos^4\alpha}}$$

$$=2\int_0^{\sqrt{2}\sin\gamma}\frac{(1-x^2)x^2dx}{\sqrt{x^4-2x^2+\sin^2 2\gamma}}.$$

But B is also $=\int_0^{\sqrt{2}\sin\gamma}\frac{\sin^2 2\gamma-x^2-(1-x^2)x^2}{\sqrt{x^4-2x^2+\sin^2 2\gamma}}\,dx$ from the expression before partial integration. Multiplying the latter expression by 2 and adding to the former,

$$3B=2\int_0^{\sqrt{2}\sin\gamma}\frac{\sin^2 2\gamma-x^2}{\sqrt{\sin^2 2\gamma-2x^2+x^4}}\,dx;$$

and substituting the above value for x,

$$\tfrac{3}{2}B=\frac{\sin^2 2\gamma}{\sqrt{2}\cos\gamma}\mathrm{F}^1+\sqrt{2}\cos\gamma(\mathrm{E}^1-\mathrm{F}^1),$$

$$B=\frac{2\sqrt{2}}{3}\cos\gamma[\mathrm{E}^1-\cos 2\gamma\,\mathrm{F}^1], \text{ the modulus being } \tan\gamma.$$

B may be calculated from this form by means of the tables in LEGENDRE'S 'Fonctions Elliptiques,' tom. ii. But A is not yet in a form adapted for numerical calculation.

The parameter $-2\sin^2\gamma$ of Π^1 is negative and numerically greater than the square of the modulus; therefore Π^1 falls within LEGENDRE'S second class (*op. cit.* tom. i. p. 72). Now it is shown by LEGENDRE (tom. i. p. 138) that

$$\frac{b^2\sin\theta\cos\theta}{\Delta(b,\,\theta)}[\Pi^1(n,\,c)-F^1(c)]=\frac{\pi}{2}+F^1(c)F(b,\,\theta)-E^1(c)F(b,\,\theta)-F^1(c)E(b,\,\theta).$$

In this case θ will be found to be $\frac{\pi}{2}-\gamma$, $\frac{b^2\sin\theta\cos\theta}{\Delta(b,\,\theta)}=\frac{1}{2}\sqrt{2}\,.\,\frac{\cos 2\gamma}{\cos\gamma}$, and $b=\frac{\cos\alpha}{\cos\gamma}$;

whence $$A=\pi-2\{E^1F-F^1(F-E)\},$$

where the moduli of F and E are $\frac{\cos\alpha}{\cos\gamma}$, and their amplitude $\frac{\pi}{2}-\gamma$.

From this form A may be calculated numerically.

APPENDIX C. (Added April 1877.)

Sir WILLIAM THOMSON, who was one of the referees requested by the Royal Society to report on this paper, has remarked that the subject of Part I. may also be treated in another manner.

The following note contains his solution, but some slight alterations have been made in a few places.

The axis of resultant moment of momentum remains invariable in space whatever change takes place in the distribution of the earth's mass; or, in other words, the normal to the invariable plane is not altered by internal changes in the earth.

Now suppose a change to take place so slowly that the moment of momentum round any axis of the motion of any part of the earth relatively to any other part may be neglected compared to the resultant moment of momentum of the whole*; or else suppose the change to take place by sudden starts, such as earthquakes. Then, on either supposition (except during the critical times of the sudden changes, if any), the component angular velocities of the mass relatively to fixed axes, coinciding with the positions of its principal axes at any instant, may be written down at once from the ordinary formulæ, in terms of the direction-cosines of the normal to the invariable plane with reference to these axes, and in terms of the moments of inertia round them, which are supposed to be known.

Hence we find immediately the angular velocity and direction of the motion of that line of particles of the solid which at any instant coincides with the normal to the invariable plane at the origin. This is equal and opposite to the angular velocity with which we see the normal to the invariable plane travelling through the solid, if we, moving with the solid, look upon the solid as fixed. Let, at any instant, x, y, z be the direction-cosines of the normal to the invariable plane relatively to the principal axes;

* This is equivalent to neglecting $\mathfrak{H}_1$, $\mathfrak{H}_2$, $\mathfrak{H}_3$ of Part I.; by which Sir W. THOMSON is of opinion that nothing is practically lost.

and let A, B, C be the principal moments of inertia at that instant. Let h be the constant moment of momentum (or twice the area conserved on the invariable plane).

Consider axes fixed relatively to the solid in the positions of the principal axes at any instant, but not moving with them, if they are being shifted in virtue of changes in the distribution of portions of the solid.

The component angular velocities of the rest of the universe are, relatively to these axes, $\frac{hx}{A}, \frac{hy}{B}, \frac{hz}{C}$; and therefore, if N be the point in which the normal to the invariable plane at the origin cuts a sphere of unit radius, the components parallel to these axes of the velocity of N relatively to them are

$$yz\left(\frac{h}{C}-\frac{h}{B}\right),\ zx\left(\frac{h}{A}-\frac{h}{C}\right),\ xy\left(\frac{h}{B}-\frac{h}{A}\right)\ *.$$

Now, suppose that by slow continuous erosion and deposition the positions of the principal axes change slowly and continuously relatively to the solid.

Let ϖ, ϱ, σ be the components round the axes (which, of course, are always mutually at right angles) of the angular velocity of the actual solid relatively to an ideal solid moving with the principal axes†. Then the component velocities relatively to this ideal solid of the point of the body coinciding at any instant with N are

$$z\varrho-y\sigma,\ x\sigma-z\varpi,\ y\varpi-x\varrho;$$

and the components parallel to the principal axes of the velocity of N relatively to these axes are $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$. Hence we have

$$\left.\begin{aligned}\frac{dx}{dt}&=yz\left(\frac{h}{C}-\frac{h}{B}\right)-(z\varrho-y\sigma),\\ \frac{dy}{dt}&=zx\left(\frac{h}{A}-\frac{h}{C}\right)-(x\sigma-z\varpi),\\ \frac{dz}{dt}&=xy\left(\frac{h}{B}-\frac{h}{A}\right)-(y\varpi-x\varrho).\end{aligned}\right\}\quad\ldots\ldots\quad(1)\ ‡$$

These three equations give $x\frac{dx}{dt}+y\frac{dy}{dt}+z\frac{dz}{dt}=0$, and therefore they are equivalent to two independent equations to determine two of the three unknown quantities x, y, z as functions of t, the three fulfilling the condition $x^2+y^2+z^2=1$, and it being understood that ϖ, ϱ, σ are given functions of the time.

To apply these equations to the questions proposed as to the earth's axis, let the normal to the invariable plane be very nearly coincident with the axis of greatest moment

* The angular velocity of the rest of the universe relatively to the earth being opposite to the angular velocity of the earth relatively to the rest of the universe, the components of the former round the axes x, y, z are taken as in the negative direction, *i.e.* from z to y, x to z, y to x.

† ϖ, ρ, σ are the same as $-\alpha$, $-\beta$, $-\gamma$ of Part I.

‡ These equations are the same as those given by me in Part I. p. 277.

of inertia C. Let O be the point where the axis C cuts the earth's surface, and let OX, OY be parallel to the axes of A and B. Then $z=1$; and if the earth's radius be taken as unity, x and y will be the coordinates relatively to OX, OY of the point P in which the normal to the invariable plane cuts the surface.

Putting therefore $z=1$ in the preceding equations, we find for the determination of x, y that

$$\left.\begin{aligned}\frac{dx}{dt}&=-yh\left(\frac{1}{\mathrm{B}}-\frac{1}{\mathrm{C}}\right)-\varrho+y\sigma,\\ \frac{dy}{dt}&=\quad xh\left(\frac{1}{\mathrm{A}}-\frac{1}{\mathrm{C}}\right)+\varpi-x\sigma,\end{aligned}\right\} \quad \ldots \ldots \quad (2)$$

or

$$\frac{dx}{dt}+ay=u,$$

$$\frac{dy}{dt}-bx=v,$$

where

$$a=h\left(\frac{1}{\mathrm{B}}-\frac{1}{\mathrm{C}}\right)-\sigma,\quad b=h\left(\frac{1}{\mathrm{A}}-\frac{1}{\mathrm{C}}\right)-\sigma \qquad \ldots \ldots \quad (3)$$

and

$$u=-\varrho,\quad v=\varpi.$$

In these equations we are to regard a, b, u, v as given functions of the time.

Eliminating y, we have

$$\frac{d}{dt}\left(\frac{1}{a}\frac{dx}{dt}\right)+bx=\frac{d}{dt}\left(\frac{u}{a}\right)-v, \quad \ldots \ldots \quad (4)$$

which is a linear equation, from which x may be found by integration; and then, by the first of equations (3),

$$y=\frac{1}{a}\left(u-\frac{dx}{dt}\right). \quad \ldots \ldots \quad (5)$$

If B=A, the presence of σ in the equations would merely mean that the axes of x and y revolve with an angular velocity σ; and so we lose nothing of interest with reference to the terrestrial problem by supposing $\sigma=0$. If, then, we take A and B constant, equation (4) becomes,

$$\frac{d^2x}{dt^2}+\omega^2x=\frac{du}{dt}-av,$$

where

$$\omega^2=ab. \qquad \ldots \ldots \quad (6)$$

To integrate this according to the method of variation of parameters, put

$$x=\mathrm{P}\cos\omega t+\mathrm{Q}\sin\omega t \quad \ldots \ldots \quad (7)$$

and

$$\frac{dx}{dt}=-\mathrm{P}\omega\sin\omega t+\mathrm{Q}\omega\cos\omega t. \quad \ldots \ldots \quad (8)$$

so that

$$\frac{dP}{dt}\cos\omega t+\frac{dQ}{dt}\sin\omega t=0.$$

We find then

$$\left.\begin{aligned}P&=-\frac{1}{\omega}\int\left(\frac{du}{dt}-av\right)\sin\omega t\,dt,\\ Q&=\ \ \frac{1}{\omega}\int\left(\frac{du}{dt}-av\right)\cos\omega t\,dt,\end{aligned}\right\}\quad\ldots\ldots\ldots\quad(9)$$

For the case considered in Part I., where u and v are constant,

$$P=-\frac{av}{\omega^2}\cos\omega t+C,\quad Q=-\frac{av}{\omega^2}\sin\omega t+C',$$

and therefore by (7)

$$x=-\frac{av}{\omega^2}+C\cos\omega t+C'\sin\omega t\ \ldots\ldots\ldots\quad(10)$$

The solution expressed in equations (5), (7), (8), (9) is convenient for discontinuous as well as for continuously varying and constant values of u and v.

Consider, then, the case of $u=0$ and $v=0$, except at certain instants when u and v have infinite values, so that $\int_{T'}^{T}udt$ and $\int_{T'}^{T}vdt$ express the components of a single abrupt change in the position of the instantaneous axis; where T and T′ denote any instants before and after the instant of the change, but so that the interval does not include more than one abrupt change.

Therefore, if t_0 be the instant of the change

$$\left.\begin{aligned}\int_{T'}^{T}v\sin\omega t\,dt&=\sin\omega t_0\int_{T'}^{T}vdt\\ \int_{T'}^{T}v\cos\omega t\,dt&=\cos\omega t_0\int_{T'}^{T}vdt\end{aligned}\right\}\quad\ldots\ldots\ldots\quad(11)$$

Hence the part of x depending on v vanishes at the instant immediately after the abrupt change when $t=t_0$. Also we have by integration by parts,

$$\left.\begin{aligned}\int\frac{du}{dt}\sin\omega t\,dt&=u\sin\omega t-\omega\int u\cos\omega t\,dt,\\ \int\frac{du}{dt}\cos\omega t\,dt&=u\cos\omega t+\omega\int u\sin\omega t\,dt\end{aligned}\right\}\quad\ldots\ldots\quad(12)$$

And, therefore, taking the integrals between the prescribed limits, since $u=0$ both when $t=T$ and when $t=T'$, we have

$$\left.\begin{aligned}\int\frac{du}{dt}\sin\omega t\,dt&=-\omega\cos\omega t_0\int_{T'}^{T}udt,\\ \int\frac{du}{dt}\cos\omega t\,dt&=\ \ \omega\sin\omega t_0\int_{T'}^{T}udt\end{aligned}\right\}\quad\ldots\ldots\quad(13)$$

Using these in (9) and (7) we find, at the instant after the abrupt change,

$$x=\int_{T'}^{T} udt \quad \quad (14)$$

and similarly

$$y=\int_{T'}^{T} vdt, \quad \quad (15)$$

which of course might be deduced from (8) and (5).